The long exact sequence of cohomology

Let
$$K: \quad \cdots \xrightarrow{d} K^i \xrightarrow{d} K^{i+1} \xrightarrow{d} \cdots$$
$$L: \quad \cdots \xrightarrow{d} L^i \xrightarrow{d} L^{i+1} \xrightarrow{d} \cdots$$
$$M: \quad \cdots \xrightarrow{d} M^i \xrightarrow{d} M^{i+1} \xrightarrow{d} \cdots$$
be cochain complexes. Additionally suppose that $0 \to K \xrightarrow{f} L \xrightarrow{g} M \to 0$ is an exact triple. In particular, for each n, the sequence $0 \to K^n \xrightarrow{f} L^n \xrightarrow{g} M^n \to 0$ is exact, and we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K^n \arrow[r, "f"] \arrow[d, "d"'] & L^n \arrow[r, "g"] \arrow[d, "d"'] & M^n \arrow[r] \arrow[d, "d"'] & 0 \\
0 \arrow[r] & K^{n+1} \arrow[r, "f"] & L^{n+1} \arrow[r, "g"] & M^{n+1} \arrow[r] & 0
\end{tikzcd}$$

Note that the rows of this diagram are exact.
We wish to construct a map $\delta: H^n(M) \to H^{n+1}(K)$.
Let x be a cocycle in M^n (i.e. $d(x)=0$). As $g: L^n \to M^n$ is surjective, there is some $\bar{x} \in L^n$ such that $g(\bar{x}) = x$. Now $gd(\bar{x}) = dg(\bar{x}) = d(x) = 0$, so $d(\bar{x}) \in \ker g: L^{n+1} \to M^{n+1}$. This implies that there is a unique $y \in K^{n+1}$ such that $f(y) = d(\bar{x})$. Furthermore, $fd(y) = df(y) = d^2(\bar{x}) = 0$, which implies that y is a cocycle. Define the map $\delta: H^n(M) \to H^{n+1}(K)$ by δ: class of $x \mapsto$ class of y. We need to show that this map is well defined, i.e:

(1) Independent of the choice of $\bar{x}$.

(2) Independent of the choice of representative of the class of x in $H^n(M)$.

(1) Suppose $\bar{x}' \in L^n$ is such that $g(\bar{x}') = x$. Then $\bar{x}' = \bar{x} + f(k)$ for some $k \in K^n$. So $d(\bar{x}') = d(\bar{x} + f(k)) = d(\bar{x}) + df(k) = f(y) + f(dk)$. Thus, choosing $\bar{x}'$ instead of $\bar{x}$ (in determining δ) maps the class of x to the class of $y + dk$ in $H^{n+1}(K)$. But the class of $y+dk$ is the same as the class of y.

(2) Suppose x' is a representative of the class of x in $H^n(M)$. Then $x' = x + dm$, for some $m \in M^{n-1}$. ~~Thus~~ There is also some $l \in L^{n-1}$ such that $g(l) = m$. So $g(\bar{x} + dl) = g(\bar{x}) + gd(l) = x + dg(l)$ $= x + dm = x'$. Now $d(\bar{x} + dl) = d(\bar{x})$, combining this with (1) implies that x and x' determine the same class in $H^{n+1}(K)$.

It is straightforward to verify that the maps $f: K^n \to L^n$ and $g: L^n \to M^n$ give well defined maps $f^*: H^n(K) \to H^n(L)$ and $g^*: H^n(L) \to H^n(M)$. Thus, we have a sequence of maps:

$$\cdots \xrightarrow{\delta} H^{n+1}(K) \xrightarrow{f^*} H^{n+1}(L) \xrightarrow{g^*} H^{n+1}(M) \xrightarrow{\delta} H^{n+2}(K) \xrightarrow{f^*} \cdots$$

prop This sequence is exact.

proof <u>Exactness at f^*</u>:

Let y be a cocycle in K^{n+1} such that the class of y is in $\operatorname{Im} \delta: H^n(M) \to H^{n+1}(K)$. Then by the construction of δ, there is ~~some~~ $\bar{x} \in L^n$ such that $d(\bar{x}) = f(y)$. So $\operatorname{Im} \delta \subset \ker f^*$. Conversely, suppose y is a cocycle in K^{n+1} such that $f^*(y) = 0$. That is, $f(y) = d(\bar{x})$ for some $\bar{x} \in L^n$.

Then $d\mathfrak{z}(\bar{x}) = \mathfrak{z}d(\bar{x}) = \mathfrak{z}f(y) = 0$. Hence $\mathfrak{z}(\bar{x})$ is a cocycle in M^n. Consequently, applying the definition of δ gives that $\delta\mathfrak{z}(\bar{x}) = y$. So $\ker f^* \subset \operatorname{Im}\delta$.

<u>Exactness at $\mathfrak{z}^*$</u>

Let x be a cocycle in L^{n+1} such that $x = f(k)$ for some $k \in K^{n+1}$. Then $\mathfrak{z}(x) = \mathfrak{z}f(k) = 0$. Hence, $\operatorname{Im} f^* \subset \ker \mathfrak{z}^*$. Conversely, if x is a cocycle in L^{n+1} such that $\mathfrak{z}^*(x) = 0$, then there is $\bar{x} \in M^n$ such that $d(\bar{x}) = \mathfrak{z}(x)$. So there is $y \in L^n$ satisfying $\mathfrak{z}(y) = \bar{x}$. Now $\mathfrak{z}d(y) = d\mathfrak{z}(y) = d(\bar{x}) = \mathfrak{z}(x)$. So $x - dy \in \ker \mathfrak{z} : L^{n+1} \to M^{n+1}$. Consequently the class of $f^*(k)$ = the class of x in $H^{n+1}(L)$. Thus, $\ker \mathfrak{z}^* \subset \operatorname{Im} f^*$.

<u>Exactness at δ:</u>

Let $\bar{x} \in L^{n+1}$ be a cocycle. By the definition of δ we have that δ: class of $\mathfrak{z}(\bar{x}) \longmapsto 0$. So $\operatorname{Im}\mathfrak{z}^* \subset \ker\delta$. Finally, suppose x is a cocycle in M^{n+1} such that $\delta(x) = 0$ in $H^{n+2}(K)$. Let $\bar{x} \in L^{n+1}$ be such that $\mathfrak{z}(\bar{x}) = x$ and let $y \in K^{n+2}$ be such that $f(y) = d(\bar{x})$ (see the construction of δ). Then y is a representative of the class of $\delta(x)$. As $\delta(x) = 0$ in $H^{n+2}(K)$, there is some $k \in K^{n+1}$ such that $d(k) = y$. Now $\mathfrak{z}(\bar{x} - f(k)) = x$. Further, $d(\bar{x} - f(k)) = d(\bar{x}) - df(k) = d(\bar{x}) - fd(k) = d(\bar{x}) - f(y) = 0$. Consequently, $\ker\delta \subset \operatorname{Im}\mathfrak{z}^*$.