EQUIVARIANT PERVERSE SHEAVES

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Contents

1. Notations and conventions

1.1. A variety will always mean a separated scheme of finite type over **C**. For a variety X, we write $D^b_c(X)$ for the bounded derived category of constructible complexes of sheaves (of k-vector spaces) with respect to the classical (= complex analytic) topology on X. We write $\mathcal{P}(X) \subset D^b_c(X)$ for the abelian category of perverse sheaves (middle perversity) on X. All operations on sheaves will be assumed to be derived. That is, we will write f_* instead of Rf_*, $\otimes$ instead of $\otimes^L$ and so on. A t-exact functor between derived categories of sheaves will always mean t-exact with respect to the perverse t-structure.

1.2. We denote by $\mathcal{D}^{\leq 0}(X)$ (resp. $\mathcal{D}^{\geq 0}$) the full subcategory of $\mathcal{D}^b_c(X)$ consisting of objects $K \in \mathcal{D}^b_c(X)$ such that there exists a stratification (depending on K), with strata $\{X_w\}_{w\in W}$, such that K is constructible with respect to this stratification, $H^j(i_w^* K) = 0$ for all $j > -\dim X_w$ and $H^j(i_w^! K) = 0$ resp. $j < -\dim X_w$, where $i_w \colon X_w \hookrightarrow X$ is the inclusion map. This is the perverse t-structure on $D^b_c(X)$. In particular, $\mathcal{P}(X) = D^{\leq 0}(X) \cap D^{\geq 0}(X)$.

1.3. An algebraic group will always mean a smooth linear algebraic group over **C**. A reductive group will always mean a connected smooth linear algebraic group over **C** with trivial unipotent radical.

1.4. The shift functor in a triangulated category will be denoted by [1].

2. Preliminaries on equivariant sheaves

2.1. Let G be an algebraic group, X a variety on which G acts on the left. Let $m\colon G\times G \to G$ denote the multiplication and let $e\colon X \to G\times X$, $x \mapsto (1, x)$ be the identity section. Let $a\colon G\times X \to X$ denote the action map and write $p_2\colon G\times X \to X$ for the projection map. A *G-equivariant* complex is a pair (K, ϕ) with $K \in D^b_c(X)$ and ϕ an isomorphism

$$\phi\colon a^* K \xrightarrow{\sim} p_2^* K \tag{2.1.1}$$

satisfying the following conditions:

(i) *Cocycle condition*: the following holds over $G \times G \times X$

$$(m \times \mathrm{id}_X)^*(\phi) = p_{23}^*(\phi) \circ (\mathrm{id}_G \times a)^*(\phi), \tag{2.1.2}$$

where $p_{23}\colon G \times G \times X \to G \times X$ is projection on the second and third factor.

(ii) *Rigidity condition*: $e^*(\phi) = \mathrm{id}_K$.

2.2. *Remark.* Let ϕ be an isomorphism as in (2.1.1) not necessarily satisfying the cocycle or the rigidity conditions. Then ϕ can be modified to make it satisfy the rigidity condition. Namely, replace ϕ by $\phi \circ a^*e^*(\phi^{-1})$. As

$$e^*(\phi \circ a^*e^*(\phi^{-1})) = e^*(\phi) \circ (ae)^*e^*(\phi^{-1}) = e^*(\phi) \circ e^*(\phi^{-1}) = \mathrm{id}_K,$$

the isomorphism $\phi \circ a^*e^*(\phi^{-1})\colon a^*K \xrightarrow{\sim} p_2^*K$ is rigid.

2.3. Morphisms of G-equivariant complexes are defined in the obvious way: let (K_1, ϕ_1), (K_2, ϕ_2) be G-equivariant complexes. A morphism $\psi\colon (K_1, \phi_1) \to (K_2, \phi_2)$ is a morphism $\psi\colon K_1 \to K_2$ such that $\phi_2 \circ a^*(\psi) = p_2^*(\psi) \circ \phi_1$. Let $\mathcal{P}_G(X)$ denote the category of G-equivariant perverse sheaves on X. That is, an object of $\mathcal{P}_G(X)$ is a G-equivariant complex (K, ϕ) such that $K \in \mathcal{P}(X)$. Let

$$\mathrm{For}\colon \mathcal{P}_G(X) \to \mathcal{P}(X), \qquad (K, \phi) \mapsto K$$

denote the forgetful functor.

2.4. **Lemma.** *Let X, Y be varieties and $f\colon Y \to X$ a smooth map of relative dimension d. Let $s\colon X \to Y$ be a section of f (i.e., $fs = \mathrm{id}_X$). Then $s^!f^* = s^*f^*[-2d]$.*

Proof. Let $\mathbf{D}$ denote Verdier duality. Then

$$\mathbf{D}s^*f^* = \mathbf{D} = s^*f^*\mathbf{D} = \mathbf{D}s^!f^! = \mathbf{D}s^!f^*[2d].$$

As $\mathbf{D}$ is an auto-equivalence this implies the result. □

2.5. **Proposition** ([BBD, Prop. 4.2.5, Cor. 4.2.6.2])**.** *If f is smooth of relative dimension d and the fibres of f are connected, then $f^*[d]$ is t-exact. Restricting $f^*[d]$ to $\mathcal{P}(X)$ gives a full and faithful functor $\mathcal{P}(X) \to \mathcal{P}(Y)$. The image of $\mathcal{P}(X)$ under $f^*[d]$ is an épaisse (= stable under subquotients) subcategory of $\mathcal{P}(Y)$.*

2.6. **Proposition.** *Let G be a connected algebraic group and X a variety on which G acts on the left. Then the forgetful functor $\mathcal{P}_G(X) \to \mathcal{P}(X)$ is full and faithful. Its essential image consists of perverse sheaves $K \in \mathcal{P}(X)$ such that $a^*K \simeq p_2^*K$.*

Proof. Let us first prove the assertion about the essential image of For. Let $K \in \mathcal{P}(X)$ and suppose we have an isomorphism $\phi\colon a^*K \xrightarrow{\sim} p_2^*K$. Thanks to Remark 2.2 we may assume that ϕ satisfies the rigidity condition. So to prove our claim it suffices to show that ϕ satisfies the cocycle condition. Let $f\colon G \times G \times X \to X$ be the projection map and let $s\colon X \to G \times G \times X$, $x \mapsto (1, 1, x)$. Then s is a section of f. Hence, from Prop. 2.5 we infer that $s^*[-2\dim G]$ is full and faithful when restricted to $\mathcal{P}(X)$. Now

$$s^*(m \times \mathrm{id}_X)^*(\phi) = (m \times \mathrm{id}_X \circ s)^*(\phi) = e^*(\phi) = \mathrm{id}_K$$

and

$$s^*(p_{23}^*(\phi)) \circ (\mathrm{id}_G \times a)^*(\phi) = (p_{23} \circ s)^*(\phi) \circ (\mathrm{id}_G \times a \circ s)^*(\phi) = e^*(\phi) \circ e^*(\phi) = \mathrm{id}_K.$$

As $s^*[-2\dim G]$ is faithful on $\mathcal{P}(X)$ this implies the cocycle condition.

It is clear that For is a faithful functor. So all that remains to be seen is that For is full. Let $(K, \phi), (K', \phi') \in \mathcal{P}_G(X)$. We need to show that any morphism $\psi\colon K \to K'$ in $\mathcal{P}(X)$ intertwines ϕ with ϕ'. The argument is similar to the one above: the morphism e is a section of p_2. Moreover, we have that

$$e^*(\phi' \circ a^*(\psi)) = \mathrm{id}_{K'} \circ \psi = \psi = \psi \circ \mathrm{id}_K = \psi \circ e^*(\phi) = e^*(p_2^*(\psi) \circ \phi).$$

Hence, as before, the claim follows from the fact that $e^*[-\dim G]$ is faithful on $\mathcal{P}(X)$. □

2.7. In view of Prop. 2.6, if G is connected, we will identify $\mathcal{P}_G(X)$ with its essential image in $\mathcal{P}(X)$. That is, we will regard $\mathcal{P}_G(X)$ as a full subcategory of $\mathcal{P}(X)$ and speak of $K \in \mathcal{P}(X)$ as being G-equivariant whenever $a^*K \simeq p_2^*K$.

2.8. **Proposition.** *Let G be a connected algebraic group. Then*

(i) *$\mathcal{P}_G(X)$ is abelian. Kernels and cokernels in $\mathcal{P}_G(X)$ coincide with those in $\mathcal{P}(X)$;*
(ii) *$\mathcal{P}_G(X)$ is an épaisse subcategory of $\mathcal{P}(X)$. That is, if $K \in \mathcal{P}_G(X)$, then every subquotient of K is also in $\mathcal{P}(X)$.*

Proof. (i) follows from the fact that both $a^*[\dim G]$ and $p_2^*[\dim G]$ are t-exact (Prop. 2.5). To see (ii) argue as follows. Let $K \in \mathcal{P}_G(X)$ and let L be a subquotient of K. Using Prop. 2.5 we infer that there is some $M \in \mathcal{P}(X)$ such that $a^*L \simeq p_2^*M$. Applying e^* gives $L \simeq M$. Hence, $a^*L \simeq p_2^*M \simeq p_2^*L$. □

2.9. **Proposition.** *Let G be a connected algebraic group, $H \subseteq G$ a closed connected subgroup. If $K \in \mathcal{P}_G(G/H)$, then K is isomorphic to a constant perverse sheaf (i.e., $K \simeq \underline{(G/H)}^{\oplus n}[\dim (G/H)]$ for some $n \in \mathbf{Z}_{\geq 0}$).*

Proof. Let $q\colon G \to G/H$ be the quotient map. Then q is smooth with connected fibres. Hence, by Prop. 2.5, $q^*[\dim H]\colon \mathcal{P}_G(G/H) \to \mathcal{P}(G)$ is t-exact, full and faithful. Further, as q is G-equivariant, the image of $\mathcal{P}_G(G/H)$ under $q^*[\dim H]$ lands in $\mathcal{P}_G(G)$. Thus, we are reduced to showing that a G-equivariant perverse sheaf on G is isomorphic to a constant perverse sheaf. Let $K \in \mathcal{P}_G(G)$, let $s\colon G \to G \times G$ be given by $g \mapsto (g, 1)$, let $\delta\colon \mathrm{pt} \hookrightarrow G$ be the inclusion of the identity element and let $c\colon G \to \mathrm{pt}$ be the obvious map. Then

$$K = (as)^*K = s^*a^*K \simeq s^*p_2^*K = (p_2 s)^*K = (\delta c)^*K = c^*\delta^*K. \qquad \square$$

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