# THE GROTHENDIECK GROUP

R. VIRK

If $\mathcal{C}$ is an abelian category, we write $K(\mathcal{C})$ for the *Grothendieck group* of $\mathcal{C}$. This is the quotient of the free abelian group with generators the objects $M \in \mathcal{C}$, by the subgroup generated by elements $M_1 - M_2 + M_3$ for every short exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

in $\mathcal{C}$. If objects in $\mathcal{C}$ have finite length and unique composition factors we write $K(\mathcal{C})^*$ for the topological dual of $K(\mathcal{C})$; i.e. for the linear functions $f : K(\mathcal{C}) \to \mathbb{Z}$ such that $f(M) = 0$ for all but finitely many isomorphism classes of irreducible objects $M \in \mathcal{C}$. If $L \in \mathcal{C}$, let's write $[L]$ for its image in $K(\mathcal{C})$. Then as $L$ runs through the irreducible objects in $\mathcal{C}$, the elements $[L]$ form a basis of $K(\mathcal{C})$, and the functions

$$\delta_L : K(\mathcal{C}) \to \mathbb{Z} \qquad \delta_L(L') = \begin{cases} 0 & \text{if } L \ncong L',\ L' \text{ irreducible} \\ 1 & \text{if } L \cong L' \end{cases}$$

form a basis of $K(\mathcal{C})^*$. More generally, if $M \in \mathcal{C}$ and $L$ is an irreducible object in $\mathcal{C}$ write $[M : L]$ for the multiplicity of $L$ in a Jordan-Holder series of $M$, and extend this bilinearly to $[\quad : \quad] : K(\mathcal{C}) \times K(\mathcal{C}) \to \mathbb{Z}$. Then for any $M \in K(\mathcal{C})$, write $\delta_M : K(\mathcal{C}) \to \mathbb{Z}$ for the function $N \mapsto [N : M]$. Now, if $F : \mathcal{C} \to \mathcal{C}'$ is an exact functor of abelian categories, we get an induced $\mathbb{Z}$-linear map $F : K(\mathcal{C}) \to K(\mathcal{C}')$, and we can define its transpose $F^* : K(\mathcal{C}')^* \to K(\mathcal{C})^*$ by $F^* f = fF$. Write $K(\mathcal{C})_{\mathbb{Q}} = K(\mathcal{C}) \otimes \mathbb{Q}$. As $K(\mathcal{C})$ is a torsion free $\mathbb{Z}$-module, $K(\mathcal{C})_{\mathbb{Q}}$ is a $\mathbb{Q}$-vector space with distinguished sublattice $K(\mathcal{C}) \subset K(\mathcal{C})_{\mathbb{Q}}$.

**Proposition 0.0.1.** *Let $F : \mathcal{C}_1 \to \mathcal{C}_2$ and $G : \mathcal{C}_2 \to \mathcal{C}_1$ be mutually adjoint exact functors between two Artinian abelian categories $\mathcal{C}_1$ and $\mathcal{C}_2$. Then $F$ and $G$ are mutual equivalences of categories if and only if they define mutually inverse isomorphisms on the level of Grothendieck groups.*

*Proof.* For $M \in \mathcal{C}_2$, let $\varphi_M$ denote the adjunction map $M \to F_1F_2(M)$. Suppose $L \in \mathcal{C}_2$ is irreducible. Since $[L] = [F_1F_2(L)]$, we have that $\varphi_L$ is an isomorphism. Now proceed by induction on length. For arbitrary $M \in \mathcal{C}_2$ we have an exact sequence $0 \to N \to M \to L \to 0$, with $N$ of lower length than $M$ and $L$ irreducible. This gives a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_N} & & \downarrow{\scriptstyle \varphi_M} & & \downarrow{\scriptstyle \varphi_L} & & \\ 0 & \longrightarrow & F_1F_2(N) & \longrightarrow & F_1F_2(M) & \longrightarrow & F_1F_2(L) & \longrightarrow & 0 \end{array}$$

The map $\varphi_N$ is an isomorphism by induction; $\varphi_L$ is an isomorphism by the previous argument. This forces $\varphi_M$ to be an isomorphism. □

Department of Mathematics, University of Wisconsin, Madison, WI 53706
*E-mail address*: virk@math.wisc.edu