HOPF ALGEBRAS

R. VIRK

0.1. **Notation.** Let k be a field. For simplicity we will assume k to be algebraically closed. We will write $\otimes$ for the usual tensor product over k. Further, for a k-vector space V, set $V^* = \mathrm{Hom}_k(V, k)$. All modules considered will be *left* modules unless otherwise stated.

0.2. **Algebras and coalgebras.** An *algebra* is a k-vector space with linear maps

$$\begin{array}{rccl} m: & A \otimes A & \to & A, \\ \imath: & k & \to & A, \end{array}$$

called the multiplication and the unit respectively, such that the following diagrams commute

$$\begin{array}{ccc} A \otimes k & \xrightarrow{\mathrm{id}\otimes\imath} & A \otimes A \\ \simeq\downarrow & & \downarrow m \\ A & \xrightarrow[\mathrm{id}]{} & A \end{array} \qquad \begin{array}{ccc} k \otimes A & \xrightarrow{\imath\otimes\mathrm{id}} & A \otimes A \\ \simeq\downarrow & & \downarrow m \\ A & \xrightarrow[\mathrm{id}]{} & A \end{array} \qquad \begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mathrm{id}\otimes m} & A \otimes A \\ m\otimes\mathrm{id}\downarrow & & \downarrow m \\ A \otimes A & \xrightarrow[m]{} & A \end{array}$$

The commutativity of the third diagram is the usual associativity axiom. Define a linear map $\sigma : A \otimes A \to A \otimes A$ by $a \otimes b \mapsto b \otimes a$. Then A is commutative if the diagram

$$\begin{array}{ccc} A \otimes A & & \\ \sigma\downarrow & \searrow^{m} & \\ A \otimes A & \xrightarrow[m]{} & A \end{array}$$

commutes.

Suppose A and B are algebras with multiplication m_A and m_B respectively. Then $A \otimes B$ is an algebra with multiplication $m_{A\otimes B}$ defined to be the composite

$$A \otimes B \otimes A \otimes B \xrightarrow{\mathrm{id}_A\otimes\sigma\otimes\mathrm{id}_B} A \otimes A \otimes B \otimes B \xrightarrow{m_A\otimes m_b} A \otimes B.$$

That is $(a \otimes b)(a' \otimes b') = (aa' \otimes bb')$.

A *coalgebra* is a k vector space with linear maps

$$\begin{array}{rcl} \Delta : A & \longrightarrow & A \otimes A, \\ \varepsilon : A & \longrightarrow & k, \end{array}$$

called the comultiplication and the counit respectively, such that the following diagrams commute

$$\begin{array}{ccc} A \otimes k & \xleftarrow{\mathrm{id}\otimes\varepsilon} & A \otimes A \\ \simeq\uparrow & & \uparrow\Delta \\ A & \xleftarrow[\mathrm{id}]{} & A \end{array} \qquad \begin{array}{ccc} k \otimes A & \xleftarrow{\varepsilon\otimes\mathrm{id}} & A \otimes A \\ \simeq\uparrow & & \uparrow\Delta \\ A & \xleftarrow[\mathrm{id}]{} & A \end{array} \qquad \begin{array}{ccc} A \otimes A \otimes A & \xleftarrow{\mathrm{id}\otimes\Delta} & A \otimes A \\ \Delta\otimes\mathrm{id}\uparrow & & \uparrow\Delta \\ A \otimes A & \xleftarrow[\Delta]{} & A \end{array}$$

The commutativity of the third diagram is referred to as coassociativity. For $a \in A$ we will use Sweedler notation and write

$$\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}.$$

Let σ be as before, a coalgebra is said to be cocommutative if the diagram

$$\begin{array}{ccc} A \otimes A & & \\ \uparrow{\scriptstyle \sigma} & \nwarrow^{\Delta} & \\ A \otimes A & \xleftarrow[\Delta]{} & A \end{array}$$

commutes.

Let A be an algebra. Identify $(A \otimes A)^*$ with $A^* \otimes A^*$. This induces a coalgebra structure on A^*, namely

$$\begin{aligned} \Delta : A^* &\to A^* \otimes A^*, \\ \varphi &\mapsto (a \otimes a' \mapsto \varphi(aa')). \end{aligned}$$

Dually, for a coalgebra C the coproduct induces an algebra structure on C^*. Namely, if $\varphi, \psi \in C^*$ then $(\varphi\psi)(x) = \sum_x \varphi(x_{(1)})\psi(x_{(2)})$, where $x \in C$.

Let C and D be coalgebras with coproduct Δ_C and Δ_D respectively. Then $C \otimes D$ is a coalgebra with comultiplication $\Delta_{C\otimes D}$ defined to be the composite

$$C \otimes D \xrightarrow{\Delta_C \otimes \Delta_D} C \otimes C \otimes D \otimes D \xrightarrow{\mathrm{id}_C \otimes \sigma \otimes \mathrm{id}_D} C \otimes D \otimes C \otimes D.$$

0.3. **Convolution product.** Suppose A is an algebra and C is a coalgebra. Identify $A \otimes C^*$ with $\mathrm{Hom}_k(C, A)$. This induces a multiplication $*$, called *convolution*, on $\mathrm{Hom}_k(C, A)$. Namely, for $\varphi, \psi \in \mathrm{Hom}_k(C, A)$

$$(\varphi * \psi)(x) = \sum_x \varphi(x_{(1)})\psi(x_{(2)}).$$

The identity of this ring is given by $\iota \circ \varepsilon$.

0.4. **Hopf algebras.** A *Hopf algebra* is a k-vector space A that is both an algebra and coalgebra such that

(i) the comultiplication Δ and the counit ε are homomorphisms of algebras;
(ii) the multiplication m and the unit $\imath$ are homomorphisms of coalgebras;
(iii) A is equipped with a bijective k-module map $S : A \to A$, called the antipode, such that the following diagrams commute

$$\begin{array}{ccc} A \otimes A & \xrightarrow{S \otimes \mathrm{id}} & A \otimes A \\ {\scriptstyle \Delta}\uparrow & & \downarrow{\scriptstyle m} \\ A & \xrightarrow[\imath \circ \varepsilon]{} & A \end{array} \qquad \begin{array}{ccc} A \otimes A & \xrightarrow{\mathrm{id} \otimes S} & A \otimes A \\ {\scriptstyle \Delta}\uparrow & & \downarrow{\scriptstyle m} \\ A & \xrightarrow[\imath \circ \varepsilon]{} & A \end{array}$$

Proposition 0.1. *The antipode is an algebra and coalgebra antiautomorphism, i.e.* $S(ab) = S(b)S(a)$ *and* $\sum_a S(a_{(2)}) \otimes S(a_{(1)}) = \sum_{S(a)} S(a)_{(1)} \otimes S(a)_{(2)}$.

Proof. Consider $\operatorname{Hom}(A\otimes A, A)$ as an algebra under the convolution product. Let $M, S', S'' \in \operatorname{Hom}(A \otimes A, A)$ be the maps given by

$$M(a \otimes a') = aa', \qquad S'(a \otimes a') = S(a')S(a) \qquad \text{and} \qquad S''(a \otimes a') = S(aa').$$

Then

$$\begin{aligned}
M * S'(x \otimes y) &= \sum_{x\otimes y} M((x \otimes y)_{(1)})S'((x \otimes y)_{(2)}) \\
&= \sum_{x,y} M(x_{(1)} \otimes y_{(1)})S'(x_{(2)} \otimes y_{(2)}) \\
&= \sum_{x,y} x_{(1)}y_{(1)}S(y_{(1)})S(x_{(2)}) \\
&= \varepsilon(x)\varepsilon(y).
\end{aligned}$$

Similarly

$$\begin{aligned}
S'' * M(x \otimes y) &= \sum_{x\otimes y} S''((x \otimes y)_{(1)})M((x \otimes y)_{(2)}) \\
&= \sum_{x,y} S''(x_{(1)} \otimes y_{(1)})M(x_{(2)} \otimes y_{(2)}) \\
&= \sum_{x,y} S(x_{(1)}y_{(1)})(x_{(2)}y_{(2)}) \\
&= \sum_{xy} S((xy)_{(1)})(xy)_{(2)} \\
&= \varepsilon(xy)
\end{aligned}$$

Now consider $\operatorname{Hom}(A, A \otimes A)$ as an algebra under convolution. Let

$$\begin{aligned}
\Delta * S' &= \sum_a (a_{(1)} \otimes a_{(2)})(S(a_{(4)}) \otimes S(a_{(3)})) \\
&= \sum_a a_{(1)}S(a_{(4)}) \otimes a_{(2)}S(a_{(3)}) \\
&= \sum_a a_{(1)}S(a_{(3)}) \otimes \varepsilon(a_{(2)}) \\
&= \sum_a a_{(1)}S(\varepsilon(a_{(2)})a_{(3)}) \otimes 1 \\
&= \sum_a a_{(1)}S(a_{(2)}) \otimes 1 \\
&= \varepsilon(a) \otimes 1.
\end{aligned}$$

Similarly

$$\begin{aligned} S'' * \Delta &= \sum_a \Delta(S(a_{(1)}))\Delta(a_{(2)}) \\ &= \sum_a \Delta(S(a_{(1)})a_{(2)}) \\ &= \Delta(\varepsilon(a)) \\ &= \varepsilon(a) \otimes 1 \end{aligned}$$

□

0.5. **Some representation theory.** Let A be a Hopf algebra throughout. We turn the field k into an A-module via

$$a \cdot 1 = \varepsilon(a), \qquad a \in A.$$

This is the *trivial representation* and by an abuse of notation is also denoted by k. The *adjoint representation* of A on itself is given by

$$\begin{aligned} \mathrm{ad} : A \otimes A &\to A, \\ a \otimes a' &\mapsto \sum_a a_{(1)} a' S(a_{(2)}). \end{aligned}$$

The *regular representation* of A on itself is given by the multiplication of A. Let V be an A-module, then V^* is also an A-module via

$$a \cdot f(v) = f(S(a)v), \qquad a \in A, v \in V, f \in V^*.$$

Furthermore, given A-modules V and W, $V \otimes W$ is also an A-module via

$$a \cdot (v \otimes w) = \sum_a a_{(1)} v \otimes a_{(2)} w, \qquad a \in A, v \in V, w \in W.$$

We also give $\mathrm{Hom}_k(U, V)$ an A-module structure through the identification

$$V \otimes U^* \simeq \mathrm{Hom}_k(U, V) \quad \text{given by} \quad v \otimes u^* \mapsto (f : u' \mapsto u^*(u')v).$$

We note that it is implicit in this statement that for a module V, V^* will always be the restricted dual of V (i.e. functions with finite dimensional support).

Warning. In general $V \otimes U^*$ is *not* isomorphic to $U^* \otimes V$ as an A-module.

Let M be an A module. The *invariants* of M are elements of the submodule

$$M^A = \{m \in M \mid am = \varepsilon(a)m \text{ for all } a \in A\}.$$

Remark 0.2. Taking invariants is a left exact functor on the category of A-modules. Further, this functor is representable, namely $M^A \simeq \mathrm{Hom}_A(k, M)$.

Lemma 0.3. *Let M, N be A-modules. Then*

$$\mathrm{Hom}_A(M, N) = \mathrm{Hom}_k(M, N)^A.$$

Proof. Suppose $\varphi \in \mathrm{Hom}_A(M,N)$, then

$$\begin{aligned}(a\varphi)(m) &= \sum_a a_{(1)}\varphi(S(a_{(2)})m)\\ &= \sum_a a_{(1)}S(a_{(2)})\varphi(m)\\ &= \varepsilon(a)\varphi(m).\end{aligned}$$

Conversely, suppose $\varphi \in \mathrm{Hom}_k(M,N)^A$ and $a \in A$ then

$$\begin{aligned}a\varphi(m) &= \sum_a a_{(1)}\varepsilon(a_{(2)})\varphi(m)\\ &= \sum_a a_{(1)}\varphi(\varepsilon(a_{(2)})m)\\ &= \sum_a a_{(1)}\varphi(S(a_{(2)})a_{(3)}m)\\ &= \sum_a (a_{(1)}\varphi)(a_{(2)}m)\\ &= \sum_a \varepsilon(a_{(1)})\varphi(a_{(2)}m)\\ &= \sum_a \varphi(\varepsilon(a_{(1)})a_{(2)}m)\\ &= \varphi(am).\end{aligned}$$

□

Write $\mathrm{Ext}_A(M,-)$ for the right derived functors of $\mathrm{Hom}_A(M,-)$.

Corollary 0.4. *Let M,N be A-modules, then there is a natural isomorphism*

$$\mathrm{Ext}^i_A(M,N) \simeq \mathrm{Ext}^i_A(k, \mathrm{Hom}_k(M,N))$$

Proof. We have the following isomorphism of functors

$$\mathrm{Hom}_A(M,-) \simeq \mathrm{Hom}_k(M,-)^A \simeq \mathrm{Hom}_A(k, \mathrm{Hom}_k(M,-)).$$

Thus, the corresponding derived functors are isomorphic. □

0.6. **An example: Group algebras.** Let G be a finite group and let $\mathbb{C}[G]$ be its group algebra over $\mathbb{C}$. Then $A = \mathbb{C}[G]$ is a Hopf algebra with

$$\Delta(g) = g \otimes g, \qquad S(g) = g^{-1} \qquad \text{and} \qquad \varepsilon(g) = 1, \qquad g \in G.$$

Set

$$\Omega = \frac{1}{|G|}\sum_{g\in G} g.$$

The element Ω is a central in A. Furthermore, $g\Omega = \Omega g = \Omega$ for all $g \in G$. Thus, letting k denote the trivial module, we have that $\Omega A \simeq k^{\oplus |G|}$. Furthermore, as Ω is a central idempotent, $A \simeq \Omega A \oplus (1-\Omega)A$. Hence, k is a projective module. Thus, if M is a finite dimensional A-module, then $\mathrm{Ext}^i(k,M) = 0$ for $i > 0$. Consequently, if M and N are finite dimensional A-modules then by Corollary 0.4 we have that $\mathrm{Ext}^i(M,N) = 0$, $i > 0$. In particular all finite dimensional A-modules are completely reducible.

Remark 0.5. The above arguments generalize to $\mathbb{F}[G]$, where $|G|$ does not divide $\mathrm{char}\,\mathbb{F}$.

Department of Mathematics, University of Wisconsin, Madison, WI 53706
E-mail address: virk@math.wisc.edu