JORDAN DECOMPOSITION AND CARTAN'S CRITERION

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Let V be a finite dimensional vector space over $\mathbb{C}$. Call $A \in \operatorname{End}(V)$ *semisimple* if the minimal polynomial of A has distinct roots. Equivalently, A is semisimple if and only if A the matrix of A is diagonalizable. Consequently, the sum and product of two commuting semisimple endomorphisms is again semisimple.

Call A *nilpotent* if $A^n = 0$ for some $n \in \mathbb{Z}_{>0}$. It follows that the product of two commuting nilpotent endomorphisms is again nilpotent and by the binomial theorem the same holds for their sum.

The following gives a refinement of the Jordan canonical form of a matrix over $\mathbb{C}$.

Proposition 0.0.1 (Jordan decomposition)**.** *Let V be a finite dimensional vector space over $\mathbb{C}$, and let $A \in \operatorname{End}(V)$. Then*

(i) *There exist unique $A_s, A_n \in \operatorname{End}(V)$ such that A_s is semisimple, A_n is nilpotent, $A_sA_n = A_nA_s$ and $A = A_s + A_n$.*
(ii) *There exist polynomials $p(x), q(x) \in \mathbb{C}[x]$, without constant term such that $A_s = p(A)$ and $A_n = q(A)$.*

Proof. Let $\prod_i (x - \lambda_i)^{m_i}$ be the characteristic polynomial (or the minimal polynomial, either works) of A, the λ_i being the distinct eigenvalues of A. By the Chinese remainder theorem, there exists $p(x) \in \mathbb{C}[x]$ such that

$$p(x) \equiv 0 \mod x, \qquad p(x) \equiv \lambda_i \mod (x - \lambda_i)^{m_i} \qquad \text{for all } i.$$

Let $A_s = p(A)$ and put $V_i = \ker(A - \lambda_i)^{m_i}$. Then $V = \bigoplus_i V_i$ and $A_s|_{V_i} = \lambda_i \cdot \mathrm{id}$. Thus, A_s is semisimple. Further, $A - A_s$ is nilpotent. So (i) and (ii) follow, except for the uniqueness statement. To prove this, let $A = A_s' + A_n'$ be another decomposition with the properties of (i). From (ii) we have that $A_sA_s' = A_s'A_s$ and $A_nA_n' = A_n'A_n$. As the sum of commuting semisimple (resp. nilpotent) endomorphisms is semisimple (resp. nilpotent), we have that $A_s - A_s'$ is semisimple and $A_n' - A_n$ is nilpotent. However, $A_s - A_s' = A_n' - A_n$, and only the zero map is both semisimple and nilpotent. Thus. $A_s = A_s'$ and $A_n = A_n'$. □

Example 0.0.2. Let $V = \mathbb{C}^3$ and let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then the characteristic polynomial of A is $(x-1)^2x$. Now $1 = (x-1)^2 - (x-2)x$. We infer that $A_s = 1 - (A-1)^2 = -A^2 + 2A$, i.e. $A_s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $A_n = A^2 - A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Proposition 0.0.3 (Cartan's criterion)**.** *Let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$ such that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. Suppose $\mathrm{Tr}(xy) = 0$ for all $x, y \in \mathfrak{g}$, then each $x \in \mathfrak{g}$ is nilpotent.*

Proof. Let $x = x_s + x_n$ be the decomposition of x from the previous proposition. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of x_s. Set $E = \mathbb{Q}\text{-span}\{\lambda_1, \ldots, \lambda_n\}$. We need to show that $E = 0$, so it suffices to demonstrate that $E^* = \mathrm{Hom}(E, \mathbb{Q})$ is zero. Let $f \in E^*$, fix a basis of V such that $x_s = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ and set $\varphi = \begin{pmatrix} f(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & f(\lambda_n) \end{pmatrix}$. If $\mathrm{Tr}(x\varphi) = \sum \lambda_i f(\lambda_i) = 0$, then applying f to $\sum \lambda_i f(\lambda_i)$ we get $\sum f(\lambda_i)^2 = 0$, i.e. $f(\lambda_i) = 0$ for each i. The only problem in attempting this argument is that φ may not be in $\mathfrak{g}$.

Let $r(x) \in \mathbb{C}[x]$ be a polynomial such that $r(\lambda_i - \lambda_j) = f(\lambda_i - \lambda_j) = f(\lambda_i) - f(\lambda_j)$ (that such a polynomial exists follows from Lagrange interpolation). Let E_{ij} be the ij'th elementary matrix, then $\mathrm{ad}\,\varphi(E_{ij}) = f(\lambda_i - \lambda_j)E_{ij} = r(\lambda_i - \lambda_j)E_{ij} = r(\mathrm{ad}\,x_s)(E_{ij})$. Hence, $\mathrm{ad}\,\varphi = r(\mathrm{ad}\,x_s)$. As $\mathrm{ad}\,x_s = (\mathrm{ad}\,x)_s$, we infer that $\mathrm{ad}\,\varphi$ is a polynomial in $\mathrm{ad}\,x$. Thus, $\mathrm{ad}\,\varphi(y) \in \mathfrak{g}$ for $y \in \mathfrak{g}$. As $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, $x = \sum_k [y_k, z_k]$ for some $y_k, z_k \in \mathfrak{g}$. So, $\mathrm{Tr}(\varphi x) = \sum_k \mathrm{Tr}(\varphi[y_k, z_k]) = \sum_k \mathrm{Tr}(\mathrm{ad}\,\varphi(y_k)z_k) = 0$. □

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