

# LIMITS AND COLIMITS

R. VIRK

A useful property of adjoint functors is this: if a functor  $F^\vee$  has a left adjoint then  $F^\vee$  preserves limits (kernels, products, inverse limits). The aim of this note is to recall the definition of limits and colimits and provide a quick proof of this property.

Let  $I$  be a small category and let  $\mathcal{C}$  be any category. An  $I$ -shaped *diagram* in  $\mathcal{C}$  (or a diagram in  $\mathcal{C}$  indexed by  $I$ ) is a functor  $D : I \rightarrow \mathcal{C}$ . A morphism  $D \rightarrow D'$  of  $I$ -shaped diagrams is a natural transformation, and we have the category  $\mathcal{C}^I$  of  $I$ -shaped diagrams in  $\mathcal{C}$ . Any object  $X$  of  $\mathcal{C}$  determines the constant diagram  $\underline{X}$  that sends each object of  $I$  to  $X$  and sends each morphism of  $I$  to  $\text{id}_X$ . A *cone* of an  $I$ -shaped diagram  $D$  is an object  $X$  of  $\mathcal{C}$  together with a morphism of diagrams  $\iota : \underline{X} \rightarrow D$ . The *limit*,  $\varprojlim D$  is a universal (final) cone of  $D$ . That is, if  $f : \underline{Y} \rightarrow D$  is a cone of the diagram  $D$ , then there is a unique map  $g : \underline{Y} \rightarrow \varprojlim D$  such that  $f = \iota \circ g$ , where  $\iota : \varprojlim D \rightarrow D$  is the diagram morphism part of the data of the cone  $\varprojlim D$ .

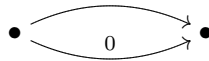
The definition of limits subsumes several constructions useful in practical settings.

**Example 0.1** (Products). If  $I$  is the discrete category on 2 points



then limits indexed by  $I$  are products of two objects.

**Example 0.2** (Kernels). Let  $I$  be the category with two objects and two parallel morphisms where one of the morphisms is a zero morphism



**Example 0.3** (Inverse limits). Let  $I$  be a partially ordered set, considered as a category by adding arrows  $a \rightarrow b$  if  $a \geq b$ . Then limits indexed by  $I$  are inverse limits.

The dual notion is that of *colimit* of a diagram  $D$ , denoted  $\varinjlim D$ . That is, a colimit is a universal cone (initial) of a diagram  $D$ . Applying the colimit construction to the above examples gives coproducts, cokernels and direct limits respectively.

*Remark 0.4.* A limit maps *to* all the objects in a big commutative diagram indexed by  $I$ . A colimit has a map *from* all the objects.

**Theorem 0.5.** *If a functor  $F^\vee : \mathcal{C} \rightarrow \mathcal{A}$  has a left adjoint, while the diagram  $D : I \rightarrow \mathcal{C}$  has a limit  $\iota : \underline{X} \rightarrow D$ , then  $F^\vee D$  has the limit  $F^\vee \iota : \underline{F^\vee X} \rightarrow F^\vee D$  in  $\mathcal{A}$ .*

*Proof.* By composition  $F^\vee \iota$  is a cone from the object  $F^\vee X$  in  $\mathcal{A}$ . Suppose we have another cone  $f : \underline{Y} \rightarrow F^\vee D$ , then applying the adjunction isomorphism we obtain a cone  $Fg : \underline{FY} \rightarrow D$  (note that  $FY$  is in  $\mathcal{C}$ ). However,  $\iota : X \rightarrow D$  is universal among cones to  $D$ , so there is a unique map  $g : \underline{FY} \rightarrow \underline{X}$  with  $\iota \circ g = Ff$ . Applying the adjunction isomorphism to  $g$  gives a unique map  $g^\# : \underline{Y} \rightarrow \underline{F^\vee X}$  with  $(F^\vee \iota) \circ g^\# = f$ .  $\square$

The dual of the theorem is equally useful: Any functor  $F$  which has a right adjoint must preserve colimits (coproducts, cokernels, direct limits).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WI 53706  
*E-mail address:* `virik@math.wisc.edu`