LIMITS AND COLIMITS

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A useful property of adjoint functors is this: if a functor $F^\vee$ has a left adjoint then $F^\vee$ preserves limits (kernels, products, inverse limits). The aim of this note is to recall the definition of limits and colimits and provide a quick proof of this property.

Let I be a small category and let $\mathcal{C}$ be any category. An I-shaped *diagram* in $\mathcal{C}$ (or a diagram in $\mathcal{C}$ indexed by I) is a functor $D : I \to \mathcal{C}$. A morphism $D \to D'$ of I-shaped diagrams is a natural transformation, and we have the category $\mathcal{C}^I$ of I-shaped diagrams in $\mathcal{C}$. Any object X of $\mathcal{C}$ determines the constant diagram $\underline{X}$ that sends each object of I to X and sends each morphism of I to id_X. A *cone* of an I-shaped diagram D is an object X of $\mathcal{C}$ together with a morphism of diagrams $\iota : \underline{X} \to D$. The *limit*, $\varprojlim D$ is a universal (final) cone of D. That is, if $f : \underline{Y} \to D$ is a cone of the diagram D, then there is a unique map $g : \underline{Y} \to \varprojlim D$ such that $f = \iota \circ g$, where $\iota : \varprojlim D \to D$ is the diagram morphism part of the data of the cone $\varprojlim D$.

The definition of limits subsumes several constructions useful in practical settings.

Example 0.1 (Products)**.** If I is the discrete category on 2 points

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then limits indexed by I are products of two objects.

Example 0.2 (Kernels)**.** Let I be the category with two objects and two parallel morphisms where one of the morphisms is a zero morphism

$$\bullet \underset{0}{\rightrightarrows} \bullet$$

Example 0.3 (Inverse limits)**.** Let I be a partially ordered set, considered as a category by adding arrows $a \to b$ if $a \geq b$. Then limits indexed by I are inverse limits.

The dual notion is that of *colimit* of a diagram D, denoted $\varinjlim D$. That is, a colimit is a universal cone (initial) of a diagram D. Applying the colimit construction to the above examples gives coproducts, cokernels and direct limits respectively.

Remark 0.4. A limit maps *to* all the objects in a big commutative diagram indexed by I. A colimit has a map *from* all the objects.

Theorem 0.5. *If a functor $F^\vee : \mathcal{C} \to \mathcal{A}$ has a left adjoint, while the diagram $D : I \to \mathcal{C}$ has a limit $\iota : \underline{X} \to D$, then $F^\vee D$ has the limit $F^\vee \iota : \underline{F^\vee X} \to F^\vee D$ in $\mathcal{A}$.*

Proof. By composition $F^\vee\iota$ is a cone from the object $F^\vee X$ in $\mathcal{A}$. Suppose we have another cone $f : \underline{Y} \to F^\vee D$, then applying the adjunction isomorphism we obtain a cone $Fg : \underline{FY} \to D$ (note that FY is in $\mathcal{C}$). However, $\iota : X \to D$ is universal among cones to D, so there is a unique map $g : \underline{FY} \to \underline{X}$ with $\iota \circ g = Ff$. Applying the adjunction isomorphism to g gives a unique map $g^\sharp : \underline{Y} \to \underline{F^\vee X}$ with $(F^\vee\iota) \circ g^\sharp = f$. $\square$

The dual of the theorem is equally useful: Any functor F which has a right adjoint must preserve colimits (coproducts, cokernels, direct limits).

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