# LOCALIZATION

R. VIRK

Let $A$ be a commutative ring (with 1). Let $S \subseteq A$ be a multiplicative set, i.e., $1 \in S$ and if $x, y \in S$, then $xy \in S$. Suppose $f : A \to B$ is a ring homomorphism satisfying

(i) $f(x)$ is a unit of $B$ for all $x \in S$;

(ii) if $g : A \to C$ is a ring homomorphism taking every element of $S$ to a unit of $C$, then there exist a unique homomorphism $h : B \to C$ such that $g = h \circ f$;

then $B$ is uniquely determined up to isomorphism, and is called the *localization* of $A$ with respect to $S$. We write $B = S^{-1}A$ and call $f : A \to S^{-1}A$ the canonical map. We prove the existence of $S^{-1}A$ as follows: define a relation $\sim$ on the set $A \times S$ by $(a, s) \sim (b, s')$ if and only if there exists $t \in S$ such that $t(s'a - sb) = 0$; it is easy to check that this is an equivalence relation (if we just have $s'a - sb = 0$ in the definition, the transitive law fails when $S$ has zero divisors). We write $\frac{a}{s}$ for the class of $(a, s)$ and define sums and products by the usual rules for calculating with fractions, i.e. $\frac{a}{s} + \frac{b}{s'} = \frac{as'+bs}{ss'}$ and $\frac{a}{s}\frac{b}{s'} = \frac{ab}{ss'}$. This makes $B$ a ring and defining $f : A \to B$ by $a \mapsto \frac{a}{1}$ we see that $f$ is a ring homomorphism satisfying the required properties. From this construction we also see that kernel of the canonical map $f : A \to S^{-1}A$ is given by $\ker(f) = \{a \in A \mid sa = 0 \text{ for some } s \in S\}$.

**Example 0.1.** Let $\mathfrak{p}$ be a prime ideal of $A$. Then $S = A - \mathfrak{p}$ is multiplicatively closed. We write $A_{\mathfrak{p}}$ for $S^{-1}A$ in this case. The elements $\frac{a}{s}$, $a \in \mathfrak{p}$ form an ideal $\mathfrak{m}$ in $A_{\mathfrak{p}}$. If $\frac{b}{t} \notin \mathfrak{m}$, then $b \notin \mathfrak{p}$, hence $b \in S$ and therefore $\frac{b}{t}$ is a unit in $A_{\mathfrak{p}}$. Consequently, $\mathfrak{m}$ is the unique maximal ideal in $A_{\mathfrak{p}}$.

*Remark* 0.2. A ring with a unique maximal ideal is called a *local ring.*

**Example 0.3.** Let $f \in A$, then $S = \{f^n\}_{n \geq 0}$ is multiplicatively closed. We write $A_f$ for $S^{-1}A$ in this case.

The construction of $S^{-1}A$ can be carried through for an $A$-module $M$ in place of the ring $A$. Define an equivalence relation $\sim$ on $M \times S$ by:

$$(m, s) \sim (m', s') \Leftrightarrow \text{ there exists } t \in S \text{ such that } t(s'm - sm') = 0.$$

Let $\frac{m}{s}$ denote the class of the pair $(m, s)$, let $S^{-1}M$ denote the set of such fractions made into a $S^{-1}A$-modile with the obvious definitions of addition and scalar multiplication. (Note that $S^{-1}M$ is also an $A$-module via the canonical map $A \to S^{-1}A$.) As in the examples before, write $M_{\mathfrak{p}}$ when $S = A - \mathfrak{p}$ and $M_f$ when $S = \{f^n\}_{n \geq 0}$.

The module $S^{-1}M$ satisfies the following universal property: suppose we are given a map $\varphi$ from $M$ to an $A$-module $N$ on which the elements of $S$ act by automorphisms. Then there is a unique map $\varphi' : S^{-1}M \to N$ such that $\varphi = \varphi' \circ \iota$.

Let $\varphi : M \to N$ be an $A$-module homomorphism. This gives rise to an $S^{-1}A$ module homomorphism $S^{-1}\varphi : S^{-1}M \to S^{-1}N$, namely $S^{-1}\varphi$ maps $\frac{m}{s}$ to $\frac{\varphi(m)}{s}$. We have that $S^{-1}(\varphi \circ \psi) = S^{-1}(\varphi) \circ S^{-1}(\psi)$ and that $S^{-1}\mathrm{id}_M = \mathrm{id}_{S^{-1}M}$. Thus, localization give a functor from the category of $A$-modules to the category of $S^{-1}A$-modules.

**Proposition 0.4.** *Localization is an exact functor. That is, if* $M' \xrightarrow{f} M \xrightarrow{g} M''$ *is exact at* $M$*, then*

$$S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} M''$$

*is exact at* $S^{-1}M$*.*

*Proof.* We have that $S^{-1}g \circ S^{-1}f = S^{-1}(f \circ g) = 0$. Hence, $\mathrm{im}\,(S^{-1}f) \subseteq \ker(S^{-1}g)$. Conversely, suppose $\frac{m}{s} \in \ker(S^{-1}g)$. Then $tg(m) = g(tm) = 0$ for some $t \in S$. So $tm \in \ker g$, hence $tm = f(m')$ for some $m' \in M'$. Thus, in $S^{-1}M$ we have $\frac{m}{s} = \frac{f(m')}{st} = (S^{-1}f)(\frac{m'}{st}) \in \mathrm{im}\,(S^{-1}f)$. Hence $\ker(S^{-1}g) \subseteq \mathrm{im}\,(S^{-1}f)$. □

**Proposition 0.5.** *Let* $M$ *be an* $A$*-module. Then the* $S^{-1}A$ *modules* $S^{-1}M$ *and* $S^{-1}A \otimes_A M$ *are isomorphic; more precisely this isomorphism is given by the map*

$$f : S^{-1}A \otimes_A M \to S^{-1}M,$$
$$\frac{a}{s} \otimes m \mapsto \frac{am}{s}.$$

*Proof.* The map $S^{-1}A \times M \to S^{-1}M$ is $A$-bilinear induces the map $f$ and is clearly surjective.

Let $\sum_i \frac{a_i}{s_i} \otimes m_i$ be any element of $S^{-1}A \otimes_A M$. Then setting $s = \prod_i s_i$ and $t_i = \prod_{j \neq i} s_i$ we have that

$$\sum_i \frac{a_i}{s_i} \otimes m_i = \sum_i \frac{a_i t_i}{s} \otimes m_i = \sum_i \frac{1}{s} \otimes a_i t_i m_i.$$

Thus, every element of $S^{-1}A \otimes_A M$ can be written in the form $\frac{1}{s} \otimes m$. Suppose $f(\frac{1}{s} \otimes m) = 0$, then $\frac{m}{s} = 0$, i.e. $tm = 0$ for some $t \in S$. Thus,

$$\frac{1}{s} \otimes m = \frac{t}{st} \otimes m = \frac{1}{st} \otimes tm = 0.$$

Hence, $f$ is injective and an isomorphism. □

**Proposition 0.6.** *Let* $M$ *be an* $A$*-module, and let* $m \in M$*. Then the following are equivalent:*

(i) $m = 0$*;*

(ii) $\frac{m}{1} = 0$ *in* $M_{\mathfrak{p}}$ *for each prime ideal* $\mathfrak{p}$ *of* $A$*;*
(iii) $\frac{m}{1} = 0$ *in* $M_{\mathfrak{m}}$ *for each maximal ideal* $\mathfrak{m}$ *of* $A$*;*

*Proof.* It is clear that (i)⇒(ii)⇒(iii). To see that (iii)⇒(i), let $m \in M$ be such that $\frac{m}{1} = 0$ in $M_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m}$ of $A$. Suppose $\mathrm{Ann}\,(m) \neq A$, let $\mathfrak{m}'$ be a maximal ideal containing $\mathrm{Ann}\,(m)$. Then we have that $sm = 0$ for some $s \in A - \mathrm{Ann}\,(m)$, which is a contradiction. Thus, $\mathrm{Ann}\,(m) = A$ and $m = 0$. □

**Corollary 0.7.** *Let* $M$ *be an* $A$*-module. Then the following are equivalent:*
(i) $M = 0$*;*
(ii) $M_{\mathfrak{p}} = 0$ *for each prime ideal* $\mathfrak{p}$ *of* $A$*;*
(iii) $M_{\mathfrak{m}} = 0$ *for each maximal ideal* $\mathfrak{m}$ *of* $A$*.*

**Proposition 0.8.** *Let* $\varphi : M \to N$ *be an* $A$*-module homomorphism. Then the following are equivalent:*
(i) $\varphi$ *is injective;*
(ii) $\varphi_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ *is injective for each prime ideal* $\mathfrak{p}$ *of* $A$*;*
(iii) $\varphi_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ *is injective for each maximal ideal* $\mathfrak{m}$ *of* $A$*.*
*Similarly with 'injective' replaced by 'surjective' throughout.*

*Proof.* From 0.4 we have that (i)⇒(ii), (ii)⇒(iii) is clear. To see (iii)⇒(i), let $M' = \ker\varphi$. Then $0 \to M' \to M \to N$ is exact, hence $0 \to M'_{\mathfrak{m}} \to M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is exact by 0.4. Thus, $M'_{\mathfrak{m}} \simeq \ker\varphi_{\mathfrak{m}} = 0$ since $\varphi_{\mathfrak{m}}$ is injective. Hence, $M' = 0$ by the previous corollary and $\varphi$ is injective.

The proof with 'injective' replaced by 'surjective' is similar. □

Department of Mathematics, University of Wisconsin, Madison, WI 53706
*E-mail address*: virk@math.wisc.edu