THE FORMALISM OF MIXED HODGE MODULES

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1. Preliminaries

In what follows 'variety' = 'scheme' = 'separated scheme of finite type over $\mathbf{C}$'. A point will always mean a closed point. By sheaf we mean a constructible sheaf of $\mathbf{C}$-vector spaces (in the analytic topology).

The terms 'functorial', 'natural' and 'canonical' will be used as synonyms for 'a morphism of functors'. For a functor F, we write $\mathbb{1}_F$ for the identity endomorphism of F.

2. Formalism of mixed Hodge modules

2.1. According to [Sa90a, §4.2], for each variety X there is an abelian category $\mathrm{MHM}(X)$, the category of *mixed Hodge modules*. Each $M \in \mathrm{MHM}(X)$ has a finite filtration W_iM, called the *weight filtration*, which is strictly compatible with any morphism in $\mathrm{MHM}(X)$, i.e., the functors $M \mapsto \mathrm{W}_iM$ and $M \mapsto \mathrm{Gr}_iM$ are exact functors for all i [Sa89, Prop. 1.5], here $\mathrm{Gr}_iM := \mathrm{W}_iM/\mathrm{W}_{i-1}M$. Furthermore, Gr_iM is semisimple for all $M \in \mathrm{MHM}(X)$ [Sa90a, §4.5].

2.2. Let $\mathrm{D}^{\mathrm{b}}_{\mathrm{mix}}(X) := \mathrm{D}^{\mathrm{b}}(\mathrm{MHM}(X))$ be the bounded derived category of $\mathrm{MHM}(X)$. By [Sa90a, Thm. 0.1] there is a faithful and exact functor

$$\mathrm{rat}\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{mix}}(X) \to \mathrm{D}^{\mathrm{b}}(X).$$

If $M \in \mathrm{D}^{\mathrm{b}}_{\mathrm{mix}}(X)$, we say that $\mathrm{rat}(M)$ is the complex of sheaves *underlying* M. Further, we will often say that a functor or morphism in $\mathrm{D}^{\mathrm{b}}_{\mathrm{mix}}(X)$ is compatible with the underlying functor/morphism in $\mathrm{D}^{\mathrm{b}}(X)$. The meaning of this is clear from the following:

2.3. By [Sa90a, (4.2.3)], there is a functor

$$\mathbf{D}\colon \mathrm{MHM}(X)^{\mathrm{op}} \to \mathrm{MHM}(X)$$

which is compatible with Verdier duality on $\mathrm{D}^{\mathrm{b}}(X)$. That is,

$$\mathrm{rat} \circ \mathbf{D} = \mathbf{D} \circ \mathrm{rat},$$

where the $\mathbf{D}$ on the right is Verdier duality. By [Sa90a, Prop. 2.6], the functor $\mathbf{D}$ *reverses weights*. That is, $\mathbf{D}\mathrm{Gr}_iM = \mathrm{Gr}_{-i}\mathbf{D}M$ and $\mathbf{D}^2M \simeq M$ canonically for all $M \in \mathrm{MHM}(X)$. Furthermore, the isomorphism $\mathbf{D}^2 \simeq \mathrm{id}$ is compatible with the underlying isomorphism in $\mathrm{D}^{\mathrm{b}}(X)$. That is, for each $M \in \mathrm{D}^{\mathrm{b}}_{\mathrm{mix}}(X)$, if $f_M\colon \mathbf{D}^2M \xrightarrow{\sim} M$ denotes the canonical isomorphism in $\mathrm{D}^{\mathrm{b}}_{\mathrm{mix}}(X)$ and $f_M^{\mathrm{rat}}\colon \mathbf{D}^2\mathrm{rat}(M) \xrightarrow{\sim} \mathrm{rat}(M)$ denotes the canonical isomorphism (Verdier duality) in $\mathrm{D}^{\mathrm{b}}(X)$, then $\mathrm{rat}(f_M) = f_M^{\mathrm{rat}}$.

2.4. Let X and Y be varieties. According to [Sa90a, (4.2.13)] there is an exact bifunctor

$$\boxtimes\colon \mathrm{MHM}(X)\times\mathrm{MHM}(Y)\to\mathrm{MHM}(X\times Y).$$

By [Sa90a, (3.8.2)], the functor $\boxtimes$ *adds weights*. That is, for $M\in\mathrm{MHM}(X), N\in\mathrm{MHM}(Y)$, we have $\mathrm{Gr}_n(M\boxtimes N)=\bigoplus_{i+j=n}\mathrm{Gr}_iM\boxtimes\mathrm{Gr}_jN$.

By [Sa90a, (2.17.4)] there is a bifunctorial isomorphism $\mathrm{rat}(M\boxtimes N)\simeq\mathrm{rat}(M)\boxtimes\mathrm{rat}(N)$ for all $M\in\mathrm{D}^{\mathrm{b}}_{\mathrm{mix}}(X), N\in\mathrm{D}^{\mathrm{b}}_{\mathrm{mix}}(Y)$. Further, [Sa90a, (2.17.4)] also implies that if Z is a third variety, then there is a trifunctorial isomorphism

$$(M\boxtimes N)\boxtimes L\simeq M\boxtimes(N\boxtimes L)$$

for all $M\in\mathrm{D}^{\mathrm{b}}_{\mathrm{mix}}(X), N\in\mathrm{D}^{\mathrm{b}}_{\mathrm{mix}}(Y), L\in\mathrm{D}^{\mathrm{b}}_{\mathrm{mix}}(Z)$. The bifunctor $\boxtimes$ and the aforementioned isomorphisms are compatible with the external tensor product on the underlying complexes of sheaves. In particular, the isomorpisms $(M\boxtimes N)\boxtimes L\simeq M\boxtimes(N\boxtimes L)$ satisfies the usual coherence law (the so called pentagon axiom) for associativity constraints. Henceforth, we will identify $(M\boxtimes N)\boxtimes L$ with $M\boxtimes(N\boxtimes L)$ and simply write $M\boxtimes N\boxtimes L$ for this object.

2.5. Let $f\colon X\to Y$ be a morphism of varieties. By [Sa90a, Thm. 4.3] there are functors

$$f_*, f_!\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{mix}}(X)\to\mathrm{D}^{\mathrm{b}}_{\mathrm{mix}}(Y)$$

that are compatible with (the derived functors of) pushforward and pushforward with proper supports, respectively. There is a natural transformation $f_!\to f_*$ which is an isomorphism if f is proper. This morphism is compatible with the underlying morphism on the corresponding functors on sheaves [Sa90a, (4.3.3)].

Furthermore, there is a natural isomorphism $\mathbf{D}f_*\simeq f_!\mathbf{D}$ which is compatible with the underlying structure on sheaves [Sa90a, (4.3.5)]. We will identify $\mathbf{D}f_*$ with $f_!\mathbf{D}$ via this isomorphism.

The functor f_* *raises weights* and the functor $f_!$ *lowers weights* [Sa90a, (4.5.2)]. That is, if M is of weight $\geq n$ (resp. $\leq n$), then f_*M (resp. $f_!M$) is also of weight $\geq n$ (resp. $\leq n$).

2.6. According to [Sa90a, §4.4], there are also functors

$$f^*, f^!\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{mix}}(Y)\to\mathrm{D}^{\mathrm{b}}_{\mathrm{mix}}(X)$$

such that f^* (resp. $f_!$) is left adjoint to f_* (resp. $f^!$). The functor f^* (resp. $f^!$) and the adjunction maps are compatible with the corresponding structures on pullback (resp. extraordinary pullback) on sheaves. As $\mathbf{D}f_*=f_!\mathbf{D}$, taking transposes we obtain that $\mathbf{D}f^*=f^!\mathbf{D}$. Using this (or adjunction) it also follows that f^* lowers weights, while $f^!$ raises weights.

Given another morphism of varieties $g\colon Y\to Z$, there are canonical isomorphisms $(gf)^*\simeq f^*g^*$ and $(gf)_!\simeq g_!f_!$ that are compatible with the underlying isomorphisms on functors on sheaves. In particular, these isomorphisms satisfy the following cocycle property: let $h\colon Z\to Z'$ be a third morphism of varieties, then the following diagrams commute

$$\begin{array}{ccc}(hgf)^* & \longrightarrow & f^*(hg)^*\\ \downarrow & & \downarrow\\ (gf)^*h^* & \longrightarrow & f^*g^*h^*\end{array}\qquad\qquad\begin{array}{ccc}(hgf)_! & \longrightarrow & (hg)_!f_!\\ \downarrow & & \downarrow\\ h_!(gf)_! & \longrightarrow & h_!g_!f_!\end{array}$$

Taking transposes, we obtain canonical isomorphisms $(gf)_*\simeq g_*f_*$ and $(gf)^!\simeq f^!g^!$ that satisfy the obvious analogue of the above cocycle property. For $?\in\{*,!\}$, we identify $(gf)^?$ (resp. $(gf)_?$) with $f^?g^?$ (resp. $g_?f_?$) via these isomorphisms.

2.7. Let flip: $X \times Y \to Y \times X$ be the isomorphism of varieties given by $(x, y) \mapsto (y, x)$. Then, by [Sa90a, (4.4.1)], there is a bifunctorial isomorphism

$$\sigma\colon \mathrm{flip}_*(M \boxtimes N) \xrightarrow{\sim} N \boxtimes M$$

for all $M \in \mathrm{D}^{\mathrm{b}}_{\mathrm{mix}}(X)$, $N \in \mathrm{D}^{\mathrm{b}}_{\mathrm{mix}}(Y)$. This isomorphism is compatible with the underlying canonical isomorphism in $\mathrm{D}^{\mathrm{b}}(X \times Y)$. In particular, $\sigma^2 = \mathrm{id}$ and σ satisfies the usual coherence law (the so called hexagon axiom) demanded of such an isomorphism.

2.8. For all $M, N \in \mathrm{D}^{\mathrm{b}}_{\mathrm{mix}}(X)$, define

$$M \otimes N := \Delta^*(M \boxtimes N) \quad \text{and} \quad \mathscr{H}om(M, N) := \Delta^!(\mathbf{D}M \boxtimes N),$$

where $\Delta\colon X \to X \times X$ is the diagonal map. By [Sa90b, Cor. 2.9] there is a trifunctorial isomorphism

$$\mathrm{Hom}(L, \mathscr{H}om(M, N)) \simeq \mathrm{Hom}(L \otimes M, N)$$

for all $L, M, N \in \mathrm{D}^{\mathrm{b}}_{\mathrm{mix}}(X)$ which is compatible with the corresponding isomorphism on the underlying objects in $\mathrm{D}^{\mathrm{b}}(X)$. Here the functor underlying $\mathscr{H}om$ is the internal sheaf Hom in $\mathrm{D}^{\mathrm{b}}(X)$.

2.9. Let X, Y, Z be varieties and $f\colon X \to Y$ a morphism. Since f can be factorized as a closed immersion (the graph map) followed by a projection, it follows from [Sa90a, (4.4.1), (4.4.2)] that there is a functorial isomorphism

$$(f \times \mathrm{id})^*(M \boxtimes N) \simeq f^*M \boxtimes N$$

for all $M \in \mathrm{D}^{\mathrm{b}}_{\mathrm{mix}}(Y)$, $N \in \mathrm{D}^{\mathrm{b}}_{\mathrm{mix}}(Z)$ which is compatible with the underlying isomorphism on sheaves.

2.10. Let $f\colon X \to Y$ be a morphism of varieties. Let $M, N \in \mathrm{D}^{\mathrm{b}}_{\mathrm{mix}}(Y)$. Let $\Delta_X\colon X \to X \times X$ and $\Delta_Y\colon Y \to Y \times Y$ be the diagonal maps. Then, by **2.8** and **2.9**,

$$\begin{aligned} f^*(M \otimes N) &= (\Delta_Y \circ f)^*(M \boxtimes N) \\ &= ((f \times f) \circ \Delta_X)^*(M \boxtimes N) \\ &\simeq \Delta_X^*(f^*M \boxtimes f^*N) \\ &= f^*M \otimes f^*N. \end{aligned}$$

This isomorphism is functorial and compatible with the underlying isomorphism in $\mathrm{D}^{\mathrm{b}}(Y)$, since all the intermediary isomorphisms are. Taking transposes we obtain a bifunctorial isomorphism

$$f_*\mathscr{H}om(f^*N, L) \simeq \mathscr{H}om(N, f_*L)$$

for all $L \in \mathrm{D}^{\mathrm{b}}_{\mathrm{mix}}(X)$, $N \in \mathrm{D}^{\mathrm{b}}_{\mathrm{mix}}(Y)$ which is compatible with the corresponding isomorphisms on the underlying objects in $\mathrm{D}^{\mathrm{b}}(X)$.

2.11. Let X, Y be varieties. By [Sa90a, Prop. 2.6, (2.17.4)] (also see [Sa90b, (2.9.3)]), we have a bifunctorial isomorphism

$$\mathbf{D}(M \boxtimes N) \simeq \mathbf{D}M \boxtimes \mathbf{D}N$$

for all $M \in \mathrm{D}^{\mathrm{b}}_{\mathrm{mix}}(X)$, $N \in \mathrm{D}^{\mathrm{b}}_{\mathrm{mix}}(Y)$ which is compatible with the underlying isomorphism in $\mathrm{D}^{\mathrm{b}}(X \times Y)$.

Let $M, N \in \mathrm{D}^{\mathrm{b}}_{\mathrm{mix}}(X)$ and let $\Delta\colon X \to X \times X$ be the diagonal map. By the above and the fact that $\mathbf{D}\Delta^* = \Delta^!\mathbf{D}$ (see **2.6**), we have

$$\begin{aligned}\mathscr{H}om(M,N) &= \Delta^!(\mathbf{D}M \boxtimes N)\\ &= \mathbf{D}\Delta^*\mathbf{D}(\mathbf{D}M \boxtimes N)\\ &\simeq \mathbf{D}\Delta^*(M \boxtimes \mathbf{D}N)\\ &= \mathbf{D}(M \otimes \mathbf{D}N).\end{aligned}$$

Further, this isomorphism is canonical and compatible with the underlying isomorphism in $\mathrm{D}^{\mathrm{b}}(X)$, since all the intermediary isomorphisms are.

2.12.

$$\begin{aligned}f^!\mathscr{H}om(M,N) &\simeq f^!\mathbf{D}(M \otimes \mathbf{D}N)\\ &= \mathbf{D}f^*(M \otimes \mathbf{D}N)\\ &\simeq \mathbf{D}(f^*M \otimes f^*\mathbf{D}N)\\ &= \mathbf{D}(f^*M \otimes \mathbf{D}f^!N)\\ &\simeq \mathscr{H}om(f^*M, f^!N).\end{aligned}$$

Taking transposes we obtain a canonical isomorphism

$$f_!M \otimes N \simeq f_!(M \otimes f^*N)$$

that is compatible with the underlying isomorphism in $\mathrm{D}^{\mathrm{b}}(X)$. Taking transposes (note the asymmetry in the isomorphism above) we also obtain a canonical isomorphism

$$f_*\mathscr{H}om(, f^!) \simeq \mathscr{H}om(f_!,)$$

2.13.

(i) Let $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ be the projection maps. Then $M \boxtimes N \simeq p^*M \otimes q^*N$ canonically for all $M \in \mathrm{D}^{\mathrm{b}}_{\mathrm{mix}}(X), N \in \mathrm{D}^{\mathrm{b}}_{\mathrm{mix}}(Y)$.

(ii) Let $\Delta_X\colon X \to X \times X$ be the diagonal map, then $M \otimes N \simeq \Delta_X^*(M \boxtimes N)$ canonically for all $M, N \in \mathrm{D}^{\mathrm{b}}_{\mathrm{mix}}(X)$.

(iii) [Sa90b, Cor. 2.9] There are trifunctorial isomorphisms $\mathrm{Hom}(L \otimes M, N) \simeq \mathrm{Hom}(K, \mathscr{H}om(M,N))$ for all $L, M, N \in \mathrm{D}^{\mathrm{b}}_{\mathrm{mix}}(X)$.

(iv) [Sa90b, (2.9.1)] There are canonical isomorphisms $\underline{\mathbf{C}}_X \otimes M \simeq M \simeq M \otimes \underline{\mathbf{C}}_X$ for each $M \in \mathrm{D}^{\mathrm{b}}_{\mathrm{mix}}(X)$.

(v) [Sa90a, §4.4] The functor f^* (resp. $f_!$) is left adjoint to f_* (resp. $f^!$).

(vi) Combining (iii), (iv) and (v) we obtain isomorphisms of functors

$$\mathrm{Hom}(-, f_*\mathscr{H}om(f^*M, N) \simeq \mathrm{Hom}(f^*-$$

(vii) [Sa90a, (4.3.2), §4.4] If $g\colon Y \to Z$ is a morphism of varieties, then there are canonical isomorphisms $(gf)_* \simeq g_*f_*$, $(gf)_! \simeq g_!f_!$, $(gf)^* \simeq f^*g^*$ and $(gf)^! \simeq f^!g^!$.

(viii) [Sa90a, (4.3.3)] There is a natural morphism $f_! \to f_*$, which is an isomorphism whenever f is *proper*.

(ix) [Sa90a, (4.4.2)] If f is *smooth of relative dimension* d, then $f^! \simeq f^*[2d](d)$.

(x) *Proper base change* [Sa90a, (4.4.3)]: Given a cartesian diagram of varieties

$$\begin{array}{ccc} X' \times_Y X & \xrightarrow{g'} & X \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & Y \end{array}$$

there is a natural isomorphism of functors $g^*f_! \simeq f'_!g'^*$.

(xi) [Sa90a, (4.5.2)] The functors $f_*, f^!$ increase weights and $f^*, f_!$ decrease weights. That is, if $\mathcal{A}^\cdot$ is of weight $\leq n$ (resp. $\geq n$), then $f_!\mathcal{A}^\cdot, f^*\mathcal{A}^\cdot$ (resp. $f_*\mathcal{A}^\cdot, f^!\mathcal{A}^\cdot$) are of weight $\leq n$ (resp. $\geq n$).

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