1. Notation

Given an algebra A, denote by $A-$mod the category of A-modules. Denote by $\mathrm{Rep}\, A$ the category of finite dimensional A-modules.

Throughout, 'functorial', 'natural' and 'canonical' will mean a morphism of functors (with the functors in question being obvious from the context). For general categorical notions the reader is referred to [KaSc].

2. Reminders

2.1. Adjunctions. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. Let (F^*, F) be an *adjoint pair* of functors, $F : \mathcal{C} \to \mathcal{D}$ and $F^* : \mathcal{D} \to \mathcal{C}$. These are the data of two natural transformations

$$\varepsilon_F : F^*F \to \mathrm{id}_{\mathcal{C}},$$
$$\eta_F : \mathrm{id}_{\mathcal{D}} \to FF^*,$$

called the *counit* and *unit* respectively, such that the compositions

$$F \xrightarrow{\eta_F \circ \mathbb{1}_F} FF^*F \xrightarrow{\mathbb{1}_F \circ \varepsilon_F} F$$

and

$$F^* \xrightarrow{\mathbb{1}_{F^*} \circ \eta_F} F^*FF^* \xrightarrow{\varepsilon_F \circ \mathbb{1}_{F^*}} F^*$$

are equal to the identity maps $\mathbb{1}_F : F \to F$ and $\mathbb{1}_{F^*} : F^* \to F^*$, respectively. Then there is an isomorphism functorial in $X \in \mathcal{C}$ and $Y \in \mathcal{D}$

$$\alpha_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(F^*Y, X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(Y, F(X)),$$
$$f \mapsto F(f) \circ \eta_F(Y).$$

The reader may verify that the inverse is given by $f' \mapsto \varepsilon_F(F^*F(X)) \circ F^*(f')$. Note that the data of such a functorial isomorphism provides a structure of an adjoint pair. Namely, set $\varepsilon_F(F^*F(X)) = \alpha^{-1}_{X,F(X)}(\mathrm{id}_{F(X)})$ and $\eta_F(Y) = \alpha_{F^*(Y),Y}(\mathrm{id}_{F^*(Y)})$.

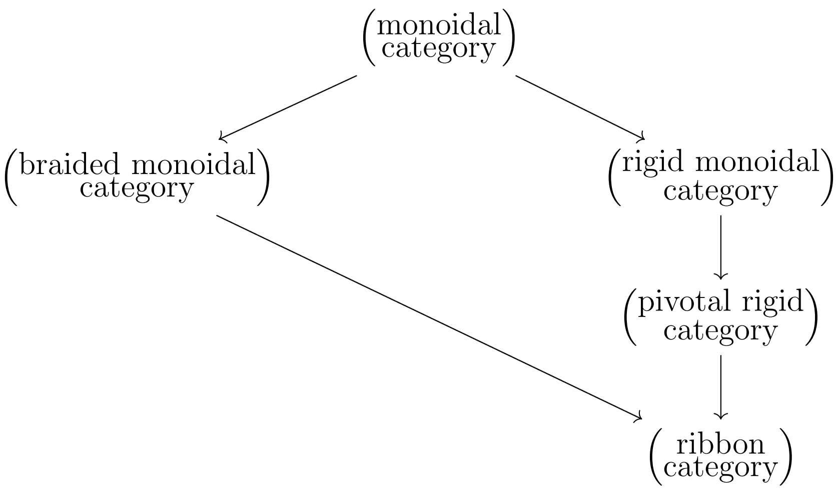
Given an adjoint pair (F^*, F), F^* is said to be *left adjoint* to F and F is said to be *right adjoint* to F^*.

3. Abstract nonsense

In this section we will define precisely the type of categories we will be working with. This avoids all possible confusion as to 'how strict' our monoidal structures are, and also makes this text essentially self contained. None of the results in this section are new. The exposition mainly follow the monographs [Ka] and [Tu] (also see [CP]).

Despite the appearance of the word 'category', the theory will have the flavor not of abstract category theory, but rather of linear algebra, due to the strong conditions we will soon impose on our categories.

We begin by drawing a diagram summarizing the relationships between the various types of monoidal categories that will appear in this text. In this diagram, an arrow from type A to type B means that the definition of type B is obtained from the definition of type A either by putting an additional structure on the category or by imposing an additional requirement.



3.1. **Monoidal categories.** A *strict monoidal category* is a 3-tuple $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1})$ consisting of a category $\mathcal{C}$, a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a distinguished object $\mathbb{1} \in \mathcal{C}$ (called the *unit object*). These data being subject to the following conditions:

(i) For all $V \in \mathcal{C}$ we have

$$\mathbb{1} \otimes V = V \qquad \text{and} \qquad V \otimes \mathbb{1} = V.$$

(ii) Suppose X_1 and X_2 are two expressions obtained from $V_1 \otimes V_2 \otimes \cdots \otimes V_m$ by inserting $\mathbb{1}$'s and parentheses: an example of such an expression is

$$(V_1 \otimes \mathbb{1}) \otimes ((V_2 \otimes V_3) \otimes \cdots \otimes V_m).$$

Then $X_1 = X_2$.

We pause to make an important technical remark. Almost all the examples of monoidal categories that arise 'in nature' are non-strict (for example, vector spaces). That is, all the equalities in the defining axioms are replaced by functorial isomorphisms. Fortunately, it is known that every monoidal category is equivalent to a strict one [MacL, Ch. XI §3, Thm. 1]. This result will constantly be invoked to omit parentheses and the associativity and unit isomorphisms in our formulas even when dealing with non-strict monoidal categories. Keeping track of these isomorphisms would only make the proofs of our results much more complicated than the ones given here and obscure the basic ideas behind the theory.

3.2. **Braiding.** A *braiding* or *R-matrix* in a strict monoidal category $\mathcal{C}$ is a collection of isomorphisms

$$\mathcal{R}_{VW} : V \otimes W \xrightarrow{\sim} W \otimes A,$$

for all $V, W \in \mathcal{C}$, satisfying the conditions:

(i) For every $f : V \to V'$ and $g : W \to W'$ in $\mathcal{C}$, the diagram

$$\begin{array}{ccc} V \otimes W & \xrightarrow{f \otimes g} & V' \otimes W' \\ {\scriptstyle \mathcal{R}_{VW}} \downarrow & & \downarrow {\scriptstyle \mathcal{R}_{V'W'}} \\ W \otimes V & \xrightarrow{g \otimes f} & W' \otimes V' \end{array} \tag{3.1}$$

commutes.

(ii) For every $V \in \mathcal{C}$, $\mathcal{R}_{V\mathbb{1}} = \mathrm{id} = \mathcal{R}_{\mathbb{1}V}$. In particular, $\mathcal{R}_{\mathbb{1}\mathbb{1}} = \mathrm{id}$.

(iii) The following two diagrams commute for all $U, V, W \in \mathcal{C}$

$$\begin{tikzcd} & U \otimes V \otimes W \arrow[dl, "\mathrm{id} \otimes \mathcal{R}_{VW}"'] \arrow[dr, "\mathcal{R}_{(U\otimes V)W}"] & \\ U \otimes W \otimes V \arrow[rr, "\mathcal{R}_{UW} \otimes \mathrm{id}"'] & & W \otimes U \otimes V \end{tikzcd} \tag{3.2}$$

and

$$\begin{tikzcd} & U \otimes V \otimes W \arrow[dl, "\mathcal{R}_{UV} \otimes \mathrm{id}"'] \arrow[dr, "\mathcal{R}_{U(V\otimes W)}"] & \\ V \otimes U \otimes W \arrow[rr, "\mathrm{id} \otimes \mathcal{R}_{UW}"'] & & V \otimes W \otimes U \end{tikzcd} \tag{3.3}$$

This gives the 'hexagon property', i.e. the diagram

$$\begin{tikzcd} & V \otimes U \otimes W \arrow[dr, "\mathrm{id} \otimes \mathcal{R}_{UW}"] & \\ U \otimes V \otimes W \arrow[ur, "\mathcal{R}_{UV} \otimes \mathrm{id}"] \arrow[rr, "\mathcal{R}_{U(V\otimes W)}"] \arrow[d, "\mathrm{id} \otimes \mathcal{R}_{VW}"'] & & V \otimes W \otimes U \arrow[d, "\mathcal{R}_{VW} \otimes \mathrm{id}"] \\ U \otimes W \otimes V \arrow[rr, "\mathcal{R}_{U(W\otimes V)}"] \arrow[dr, "\mathcal{R}_{UW} \otimes \mathrm{id}"'] & & W \otimes V \otimes U \\ & W \otimes U \otimes V \arrow[ur, "\mathrm{id} \otimes \mathcal{R}_{UV}"'] & \end{tikzcd} \tag{3.4}$$

commutes for all $U, V, W \in \mathcal{C}$. The top and bottom triangles commute by (3.2) and (3.3) and the middle rectangle commutes by (3.1). **NOT PHRASED QUITE RIGHT, BESIDES SHOW THIS**

3.3. Duals. Let $\mathcal{C}$ be a strict monoidal category and let $V \in \mathcal{C}$. A *left dual* to V is an object V^* with two morphisms

$$\varepsilon_V : V^* \otimes V \to \mathbb{1} \qquad \text{and} \qquad \eta_V : \mathbb{1} \to V \otimes V^*, \tag{3.5}$$

such that the compositions

$$V = \mathbb{1} \otimes V \xrightarrow{\eta_V \otimes \mathrm{id}} V \otimes V^* \otimes V \xrightarrow{\mathrm{id} \otimes \varepsilon_V} V \otimes \mathbb{1} = V \tag{3.6}$$

and

$$V^* = V^* \otimes \mathbb{1} \xrightarrow{\mathrm{id} \otimes \eta_V} V^* \otimes V \otimes V^* \xrightarrow{\varepsilon_V \otimes \mathrm{id}} \mathbb{1} \otimes V^* = V^* \tag{3.7}$$

are equal to the identity.

Similarly, define a *right dual* of an object V to be an object $V^{\circledast}$ with morphisms

$$\varepsilon'_V : V \otimes V^{\circledast} \to \mathbb{1} \qquad \text{and} \qquad \eta'_V : \mathbb{1} \to V^{\circledast} \otimes V, \tag{3.8}$$

satisfying the obvious analogues of the identities in the previous definition.

The terminology used here is taken from [Ka, Ch. XIV]. In [Tu], only V^* is considered, and is simply called the dual of V. Our choice of 'left/right' is justified by the following:

Lemma 3.1. *Let $V \in \mathcal{C}$ and suppose V has a left dual V^*. Then the functor $V \otimes -$ has a left adjoint given by $V^* \otimes -$. Similarly if $V^{\circledast}$ is a right dual then $V \otimes -$ has a right adjoint given by $V^{\circledast} \otimes -$.*

Proof. To $\psi \in \mathrm{Hom}_{\mathcal{C}}(V^* \otimes X, Y)$ associate the map in $\mathrm{Hom}_{\mathcal{C}}(X, V \otimes Y)$ given by the composition

$$X = \mathbb{1} \otimes X \xrightarrow{\eta_V \otimes \mathrm{id}} V \otimes V^* \otimes X \xrightarrow{\mathrm{id} \otimes \psi} V \otimes Y.$$

Similarly, to $\psi' \in \mathrm{Hom}_{\mathcal{C}}(X, V \otimes Y)$ associate the map in $\mathrm{Hom}_{\mathcal{C}_q}(V^* \otimes X, Y)$ given by the composition

$$V^* \otimes X \xrightarrow{\mathrm{id} \otimes \psi'} V^* \otimes V \otimes Y \xrightarrow{\varepsilon_V \otimes \mathrm{id}} \mathbb{1} \otimes Y = Y.$$

The above two maps are inverse to each other. It is now clear that this gives an adjunction $\mathrm{Hom}_{\mathcal{C}_q}(V^* \otimes -, -) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}_q}(-, V \otimes -)$. The proof that $V^{\circledast} \otimes -$ is right adjoint to $V \otimes -$ is similar. □

Corollary 3.2. *Let $V \in \mathcal{C}$ and suppose V has a left and right dual. Then the functor $V \otimes -$ is exact.*

Proof. By the Lemma, $V \otimes -$ admits left and right adjoints, thus it is exact (see [GeMa, Ch. II §6.20]). □

3.4. Rigid categories. A strict monoidal category $\mathcal{C}$ is called *rigid* if every object in $\mathcal{C}$ has left and right duals.

Let $\mathcal{C}$ be rigid, for $V \in \mathcal{C}$ turn the assignment $V \mapsto V^*$ into a contravariant functor as follows. Given a morphism $f : V \to W$, define $f^* : W^* \to V^*$ to be the composition

$$W^* = W^* \otimes \mathbb{1} \xrightarrow{\mathrm{id} \otimes \eta_V} W^* \otimes V \otimes V^* \xrightarrow{\mathrm{id} \otimes f \otimes \mathrm{id}} W^* \otimes W \otimes V^* \xrightarrow{\varepsilon_W \otimes \mathrm{id}} \mathbb{1} \otimes V^* = V^* \tag{3.9}$$

Lemma 3.3. *Let $\mathcal{C}$ be rigid and let $V, W \in \mathcal{C}$. Then there is a functorial isomorphism*

$$(V \otimes W)^* \simeq W^* \otimes V^*.$$

Proof. The isomorphism is given by the composition

$$\begin{aligned}(V \otimes W)^* = (V \otimes W)^* \otimes \mathbb{1} &\xrightarrow{\mathrm{id} \otimes \eta_V} (V \otimes W)^* \otimes V \otimes V^* \\ &\xrightarrow{\mathrm{id} \otimes \mathrm{id} \otimes \eta_W \otimes \mathrm{id}} (V \otimes W)^* \otimes V \otimes W \otimes W^* \otimes V^* \\ &\xrightarrow{\varepsilon_{V \otimes W} \otimes \mathrm{id} \otimes \mathrm{id}} \mathbb{1} \otimes W^* \otimes V^* = W^* \otimes V^*.\end{aligned}$$

□

Remark 3.4. For an abelian category $\mathcal{C}$, it's *Grothendieck group* $K_0(\mathcal{C})$ is the quotient of the free abelian group with generators $[V]$, $V \in \mathcal{C}$, by the subgroup generated by elements $[V'] - [V] + [V'']$ for every short exact sequence $0 \to V' \to V \to V'' \to 0$ in $\mathcal{C}$. The elements $[L]$, as L runs through isomorphism classes of irreducible objects in $\mathcal{C}$, form a basis for $K_0(\mathcal{C})$. Additionally, if objects in $\mathcal{C}$ have finite length and unique composition factors, write $[M : L]$ for the coefficient of $[L]$ when $[M]$ is expressed in terms of this basis. When the category $\mathcal{C}$ is rigid strictly monoidal we can make $K_0(\mathcal{C})$ a ring, the *Grothendieck ring* of $\mathcal{C}$, by defining $[V][W] = [V \otimes W]$. Note that we need the exactness of $\otimes$ (provided by Corollary 3.2) in order for this to be well defined.

Clearly, $K_0(\mathcal{C})$ is an associative ring with unit; if in addition $\mathcal{C}$ is braided, then $K_0(\mathcal{C})$ is commutative.

We also record the following:

Lemma 3.5. *Let $\mathcal{C}$ be rigid. If $\varphi = \{\varphi_V : V \xrightarrow{\sim} V \,|\, V \in \mathcal{C}\}$ is an automorphism of the identity functor such that $\varphi_{V\otimes W} = \varphi_V \otimes \varphi_W$, for all $V, W \in \mathcal{C}$, then*

$$\varphi_{\mathbb{1}} = \mathrm{id} \qquad \text{and} \qquad \varphi_{V^*} = (\varphi_V^*)^{-1}, \qquad V \in \mathcal{C},$$

where φ_V^* is defined by (3.9).

Proof. □

3.5. Pivotal categories. A rigid category $\mathcal{C}$ is called *pivotal* if for every $V \in \mathcal{C}$ there exist functorial isomorphisms

$$\delta_V : V \xrightarrow{\sim} V^{**}, \tag{3.10}$$

such that

$$\delta_{V\otimes W} = \delta_V \otimes \delta_W, \tag{3.11}$$

for all $V, W \in \mathcal{C}$.

Remark 3.6. As the reader may verify, a pivotal structure on $\mathcal{C}$ is equivalent to the existence of a functorial isomorphism between left and right duals.

Lemma 3.7. *Let $\mathcal{C}$ be pivotal. Then*

$$\delta_{\mathbb{1}} = \mathrm{id}, \qquad \text{and} \qquad \delta_{V^*} = (\delta_V^*)^{-1}, \qquad V \in \mathcal{C},$$

where δ_V^* is defined by (3.9).

Proof. □

3.6. Ribbon categories. A *ribbon* category $\mathcal{C}$ is a braided pivotal category equipped with an automorphism $\theta = \{\theta_V : V \xrightarrow{\sim} V \,|\, V \in \mathcal{C}\}$ of the identity functor, satisfying the following two conditions:

$$\theta_{V\otimes W} \circ (\theta_V \otimes \theta_W)^{-1} = \mathcal{R}_{WV} \circ \mathcal{R}_{VW}, \qquad \text{for all } V, W \in \mathcal{C}, \tag{3.12}$$

$$\theta_{V^*} = \theta_V^*, \qquad \text{for all } V \in \mathcal{C}. \tag{3.13}$$

The automorphism θ has various incarnations in the literature: the Casimir, ribbon element, universal twist, balancing etc. We will always refer to it as the Casimir.

Remark 3.8. It can be shown that a braided rigid category is automatically pivotal. Namely, let $\psi_V : V \xrightarrow{\sim} V^{**}$ be given by the composition

$$V = V \otimes \mathbb{1} \xrightarrow{\mathrm{id}\otimes\eta_{V^*}} V \otimes V^* \otimes V^{**} \xrightarrow{\mathcal{R}_{VV^*}\otimes\mathrm{id}} V^* \otimes V \otimes V^{**} \xrightarrow{\varepsilon_V\otimes\mathrm{id}} \mathbb{1} \otimes V^{**} = V^{**}.$$

Then $\psi_V \circ \theta_V$ defines a pivotal structure on $\mathcal{C}$.

3.7. Example: representations of $U_q(\mathfrak{sl}_2)$. Let $U_q'(\mathfrak{sl}_2)$ be the $\mathbb{Q}(q)$ algebra generated by $E, F, K^{\pm1}$ and relations:

$$KK^{-1} = 1K^{-1}K, \qquad KE = q^2EK, \qquad KF = q^{-2}FK,$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

U_q is a Hopf algebra with coproduct Δ, antipode S and counit ε given by

$$\Delta(E) = E \otimes 1 + K \otimes E, \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F, \quad \Delta(K) = K \otimes K.$$

$$S(E) = -K^{-1}E, \quad S(F) = -FK, \qquad S(K) = K^{-1}.$$

$$\varepsilon(E) = 0, \quad \varepsilon(F) = 0, \quad \varepsilon(K) = 1.$$

The inverse of the antipode is given by

$$S^{-1}(E) = -EK^{-1}, \quad S^{-1}(F) = -KF, \quad S^{-1}(K) = K^{-1}.$$

Let $M, N \in U_q-$mod and let $M \otimes N$ be the usual tensor product of vector spaces. Endow this space with a U_q-action via the algebra homomorphism $\Delta : U_q \to U_q \otimes U_q$.

Also endow the field $\mathbb{C}$ with a U_q-action via $\varepsilon : U_q \to \mathbb{C}$ and denote the resulting module by $\mathbb{1}$. This gives U_q-mod the structure of a strict monoidal category.

Let $V \in \operatorname{Rep} U_q$. Let $V^* = \operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C})$ with U_q-action given by

$$x \cdot f = \langle f, S(x)-\rangle, \qquad x \in U_q, f \in V \in \operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C}).$$

That this is a honest U_q-action is a consequence of the antipode being an algebra anti-homomorphism. Define linear maps

$$\begin{array}{rccc} \varepsilon_V : & V^* \otimes V & \to & \mathbb{1}, \\ & f \otimes v & \mapsto & \langle f, v\rangle, \end{array} \quad \text{and} \quad \begin{array}{rccc} \eta_V : & \mathbb{1} & \to & V \otimes V^*, \\ & 1 & \mapsto & \sum_i v_i \otimes v^i, \end{array}$$

where $\{v_i\}$ and $\{v^i\}$ are dual bases in V and $\operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C})$. The element $\sum_i v_i \otimes v^i$ is independent of the chosen basis. Indeed, if $\{e_i\}$ and $\{e^i\}$ is another pair of dual bases, then

$$\sum_i e_i \otimes e^i = \sum_{i,j} e_i \otimes \langle e^i, v_j\rangle v^j = \sum_{i,j} \langle e^i, v_j\rangle e_i \otimes v^j = \sum_j v_j \otimes v^j.$$

Remark 3.9. Although the above computation is trivial, that this element in $V \otimes V^*$ is independent of the basis may appear mysterious to the reader encountering it for the first time. Perhaps the following description will further elucidate this: identify the vector space $V \otimes V^*$ with $\operatorname{End}_{\mathbb{C}}(V)$ via $v \otimes f \mapsto \langle f, -\rangle v$. As V is finite dimensional this is an isomorphism of vector spaces. The element $\sum_i v_i \otimes v^i$ is now merely the image of id_V under the inverse map, which is of course independent of any chosen basis.

The reader may verify that ε_V and η_V are in fact U_q-module homomorphisms. This is a formal consequence of the axioms of a Hopf algebra, but in our case may also be verified directly from definitions. Furthermore, it is clear that this data satisfies all the requirements for a left dual.

Define $V^{\circledast}$ to be the vector space $\operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C})$ with U_q action given by

$$x \cdot f = \langle f, S^{-1}(x)-\rangle, \qquad x \in U_q, f \in \operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C}).$$

Define maps

??

where as before $\{v_i\}$ and $\{v^i\}$ are dual bases in V and $\operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C})$ respectively. That this data satisfies the requirements for a right dual follows from arguments similar to those for V^*.

Thus, $\operatorname{Rep} U_q$ is rigid.

Now a simple computation using the defining relations shows that

$$K^{-2}S^{-1}(x) = S(x)K^{-2}, \qquad \text{for all } x \in U_q.$$

Consequently, the map

$$\delta_V : V \to V^{**} \qquad \text{given by} \qquad v \mapsto \langle -, K^{-2}v\rangle$$

is a U_q-module homomorphism. That it is an isomorphism is immediate from the finite dimensionality of V. Furthermore as $\Delta(K) = K \otimes K$, we have that, for another module W in $\operatorname{Rep} U_q$, $\delta_(V \otimes W) = \delta_V \otimes \delta_W$ modulo the identification in ???.

Thus, $\operatorname{Rep} U_q$ is in fact pivotal. Furthermore, the quantum trace of an endomorphism f of V is given by

$$\mathrm{tr}_q(f) = \mathrm{tr}(K^{-2}f)\mathrm{id}_{\mathbb{1}},$$

where tr is the ordinary trace of f.

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