NAKAYAMA'S LEMMA

R. VIRK

Lemma 0.0.1. *Let M be a finitely generated module over a commutative ring A (with 1). Suppose there is an ideal of A such that $M = IM$. Then there is an element $a \in 1 + I$ such that $aM = 0$.*

Proof. Let $e_1, \dots, e_n$ be a set of generators for M (over A). Then we have that

$$\begin{aligned} e_1 &= x_{11}e_1 + x_{12}e_2 + \cdots + x_{1n}e_n, \\ e_2 &= x_{21}e_1 + x_{22}e_2 + \cdots + x_{2n}e_n, \\ \vdots \quad & \quad \ \ \vdots \qquad\quad \vdots \qquad\qquad \vdots \\ e_n &= x_{n1}e_1 + x_{n2}e_2 + \cdots + x_{nn}e_n, \end{aligned}$$

for $x_{ij} \in I$. Let X be the matrix (x_{ij}), then the above set of equations is equivalent to

$$(X - \mathrm{id}) \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix} = 0,$$

where id is the identity matrix. Multiplying by the adjoint matrix of $X - \mathrm{id}$ we obtain that

$$\det(X - \mathrm{id}) \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix} = 0,$$

where det denotes determinant. Thus, $\det(X - \mathrm{id})e_i = 0$ for all i, consequently $\det(X - \mathrm{id})M = 0$. Expanding the determinant we see that $\det(X - \mathrm{id}) = 1 + a$, with $a \in I$. □

Department of Mathematics, University of Wisconsin, Madison, WI 53706
E-mail address: virk@math.wisc.edu