The Orbit-Stabilizer Theorem

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An *action* of a group G on a set S is a map

$$G \times S \longrightarrow S$$

notated by the juxtaposition $(g, s) \mapsto gs$, such that $1_G s = s$ for all $s \in S$ and $(g_1 g_2)s = g_1(g_2 s)$ for all $g_1, g_2 \in G$ and $s \in S$. The *orbit* of a point $s \in S$ under the action is the set of points $\mathcal{O}_s = \{gs \,|\, g \in G\}$. The *stabilizer* of s is $stab(s) = \{g \in G \,|\, gs = s\}$.

Theorem (Orbit-Stabilizer)**.** $|G| = |\mathcal{O}_p||stab(p)|$

Proof. For every $x \in \mathcal{O}_p$ define

$$H_x = \{g \,|\, gp = x, \, g \in G\}$$

Clearly for distinct x, y H_x and H_y are disjoint; as if $g \in H_x$ and $g \in H_y$ we have that $gp = x$ as well as $gp = y$ which gives us that $x = y$. Furthermore as sets

$$G = \bigcup_{x \in \mathcal{O}_p} H_x$$

As clearly $\bigcup_{x \in \mathcal{O}_p} H_x \subseteq G$ and if $g \in G$ we have that $gp = s$ for some $s \in S$ which gives us that $s \in \mathcal{O}_p$ and hence $g \in H_s$, thus $G \subseteq \bigcup_{x \in \mathcal{O}_p} H_x$.

Thus we have that

$$|G| = \sum_{x \in \mathcal{O}_p} |H_x|$$

Note that $p \in \mathcal{O}_p$ and $H_p = stab(p)$. We will show that $|H_x| = |H_p|$ for all $x \in \mathcal{O}_p$ which will subsequently give us that

$$|G| = \sum_{x \in \mathcal{O}_p} |H_x| = |\mathcal{O}_p||stab(p)|$$

Pick some (fixed) $y \in H_x$ and define a map from $stab(p)$ to H_x by

$$h \longrightarrow yh$$

for $h \in stab(p)$.

We need to show that this map is a bijection. Clearly the map is injective, to see surjectivity let $h \in H_x$ and consider the element $y^{-1}h$. Now $yp = x$ as $y \in H_x$, this implies that $y^{-1}x = p$ which gives us that

$$y^{-1}hp = y^{-1}x = p$$

Thus, $y^{-1}h \in stab(p)$. Furthermore $y(y^{-1}h) = h$. So our map is surjective and we are done!

$\square$