SCHUR'S LEMMA

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Lemma 0.1. *Let V be a countable dimensional vector space over $\mathbb{C}$. If $\varphi \in \mathrm{Hom}_{\mathbb{C}}(V, V)$, then there exists $c \in \mathbb{C}$ such that $T - c \cdot \mathrm{id}$ is not invertible on V.*

Proof. Suppose that $\varphi - c \cdot \mathrm{id}$ is invertible for all $c \in \mathbb{C}$. Then $P(\varphi)$ is invertible for all non-zero polynomials P in one variable. So, if $R = P/Q$ is a rational function with P and Q polynomials, then we can define $R(\varphi) = P(\varphi)(Q(\varphi))^{-1}$. This gives us a map $\mathbb{C}(x) \to \mathrm{Hom}_{\mathbb{C}}(V, V)$. If $v \in V$ is non-zero and $R(\varphi)$ is as above, then $R(\varphi) = 0$ only if $P(\varphi) = 0$, which implies that P is the zero polynomial (as otherwise we can find an eigenvector for φ). Thus, the map $\mathbb{C}(x) \to V$ is injective. This is a contradiction since $\mathbb{C}(x)$ is of uncountable dimension over $\mathbb{C}$. □

Lemma 0.2 (Schur's lemma)**.** *Suppose that V is a countable dimension vector space over $\mathbb{C}$ and that A is an algebra that acts irreducibly on V. If $\varphi \in \mathrm{Hom}_{\mathbb{C}}(V, V)$ commutes with the action of A, then φ is a scalar multiple of the identity operator.*

Proof. By the previous lemma, there exists $c \in \mathbb{C}$, such that $\varphi - c \cdot \mathrm{id}$ is not invertible. As $\ker(\varphi - c \cdot \mathrm{id})$ is a submodule of V it is either 0 or all of V. If it is 0, then $\mathrm{im}(\varphi - c \cdot \mathrm{id})$ is all of V and $\varphi - c \cdot \mathrm{id}$ is invertible, which is a contradiction. Thus, $\varphi - c \cdot \mathrm{id} = 0$ and $\varphi = c \cdot \mathrm{id}$. □

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