The Cubic is solvable by Radicals

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Consider the general cubic

$$x^3 - ax^2 + bx - c = 0$$

let α, β, γ be roots of this equation and let

$$t = \alpha + \beta\omega + \gamma\omega^2$$

where ω is a primitive third root of unity ($\omega = \frac{-1+\sqrt{-3}}{2}$). Now let $t_1, \dots, t_6$ be the values of t under the $3! = 6$ permutations of the roots α, β, γ. Consider the polynomial

$$p(x) = (X - t_1)(X - t_2)(X - t_3)(X - t_4)(X - t_5)(X - t_6)$$

The coefficients of $p(x)$ are symmetric in $t_1, \dots, t_6$ and are hence symmetric in the α, β, γ. Thus, these coefficients can be expressed in terms of a, b, c. Hence, if we can solve the polynomial $p(x)$ we will in particular know the values of $\alpha + \beta\omega + \gamma\omega^2$, $\alpha + \beta\omega^2 + \gamma\omega$ and $\alpha + \beta + \gamma = c$, summing these three quantities we get 3α and we have thus solved for α. We can similarly solve for β and γ.

We actually seem to have increased our work here, as $p(x)$ is a degree 6 polynomial (compared to solving the original degree 3 polynomial). However, we can relabel $t_1, \dots, t_6$ so that $t_2 = \omega t_1$, $t_3 = \omega^2 t_1$; $t_5 = \omega t_4$ and $t_6 = \omega^2 t_4$. Also note that

$$(y - p)(y - \omega p)(y - \omega^2 p) = y^3 - p^3$$

Thus, with our new labelling we have that

$$p(x) = (X^3 - t_1^3)(X^3 - t_4^3)$$

which is just a quadratic in X^3 and can be solved by the usual quadratic formula. This would enable us to solve for α, β, γ by the argument given earlier, except for one snag. When we solve for the roots of $p(x)$ there is no way to identify which solution is $t_1, t_2, \dots$ etc., this is easily resolved by noting that $(\alpha + \omega\beta + \omega^2\gamma)(\alpha + \omega^2\beta + \omega\gamma)$ is symmetric in α, β, γ and thus a know quantity say m. From this it follows that the three roots of the original cubic are $[\frac{1}{3}(\alpha + \beta + \gamma) + s + \frac{m}{s}]$, $[\frac{1}{3}(\alpha + \beta + \gamma) + \omega s + \frac{m}{\omega s}]$, $[\frac{1}{3}(\alpha + \beta + \gamma) + \omega^2 s + \frac{m}{\omega^2 s}]$.