Tensor products

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Let A be a ring (commutative with unity), L, M and N three A-modules. We say that a map $\varphi : M \times N \to L$ is bilinear if fixing either of the entries it is A-linear in the other, that is if:

$$\varphi(x + x', y) = \varphi(x, y) + \varphi(x', y), \quad \varphi(ax, y) = a\varphi(x, y),$$
$$\varphi(x, y + y') = \varphi(x, y) + \varphi(x, y'), \quad \varphi(x, ay) = a\varphi(x, y).$$

Write $\mathcal{L}(M, N; L)$ or $\mathcal{L}_A(M, N; L)$ for the set of all bilinear maps from $M \times N$ to L; this has an A-module structure (since A is commutative).

If $g : L \to L'$ is an A-linear map and $\varphi \in \mathcal{L}(M, N; L)$ then $g \circ \varphi \in \mathcal{L}(M, N; L')$. For given M and N, consider a bilinear map $\otimes : M \times N \to L_0$ having the following property, where we write $x \otimes y$ instead of $\otimes(x, y)$: for any A-module L and any $\varphi \in \mathcal{L}(M, N; L)$ there exists a unique A-linear map $g : L_0 \to L$ satisfying

$$g(x \otimes y) \to \varphi(x, y).$$

If this holds we say that L_0 is the *tensor product of M and N over A*, and write $L_0 = M \otimes_A N$; we sometimes omit A and write $M \otimes N$. $M \otimes N$ assuming it exists is uniquely determined (upto isomorphism). To prove existence, write F for the free A-module with basis the set $M \times N$, and let $R \subset F$ be the submodule generated by all elements of the form

$$(x + x', y) - (x, y) - (x', y), \quad (ax, y) - a(x, y)$$
$$(x, y + y') - (x, y) - (x, y'), \quad (x, ay) - a(x, y).$$

Now set $L_0 = F/R$ and write $x \otimes y$ for the image in L_0 of $(x, y) \in F$. It follows that L_0 and $\otimes$ satisfy the condition for the tensor product.

Note that the general element of $M \otimes N$ is a sum of the form $\sum x_i \otimes y_i$, and cannot necessarily be written as $x \otimes y$.

For A-modules M, N and L the definition of the tensor product gives that:

$$Hom_A(M \otimes N, L) \cong \mathcal{L}(M, N; L). \quad (1)$$

The canonical isomorphism is obtained by taking an element φ of the right-hand side to the element g of the left-hand side satisfying $g(x \otimes y) = \varphi(x, y)$.

We can define multilinear maps from an r-fold product of A-modules $M_1, \dots, M_r$ to an A-module L just as in the bilinear case, and get modules $\mathcal{L}(M_1, \dots, M_r; L)$ and $M_1 \otimes \cdots \otimes M_r$; the following associative law then holds:

$$(M \otimes M') \otimes M'' = M \otimes M' \otimes M'' = M \otimes (M' \otimes M''). \quad (2)$$

the following also hold

$$M \otimes N \cong N \otimes M. \tag{3}$$
$$M \otimes A = M. \tag{4}$$
$$(\oplus_\lambda M_\lambda) \otimes N = \oplus_\lambda (M_\lambda \otimes N). \tag{5}$$

If $f : M \to M'$ and $g : N \to N'$ are both A-linear then $(x, y) \mapsto f(x) \otimes g(y)$ is a bilinear map from $M \times N$ to $M' \otimes N'$, and so it defines a linear map $M \otimes N \to M' \otimes N'$, which we denote $f \otimes g$. By definition we have:

$$(f \otimes g)(\sum_i x_i \otimes y_i) = \sum_i f(x_i) \otimes g(y_i). \tag{6}$$

If f and g are surjective then so is $f \otimes g$ with kernel generated by $\{x \otimes y | f(x) = 0 \text{ or } g(y) = 0\}$. To see this, let $T \subset M \otimes N$ be the submodule generated by this set; clearly $T \subseteq ker(f \otimes g)$ so that $f \otimes g$ induces a linear map $\alpha : (M \otimes N)/T \to M' \otimes N'$; furthermore, we can define a bilinear map $M' \times N' \to (M \otimes N)/T$ by

$$f(x', y') \mapsto (x \otimes y \mod T), \qquad \text{where } f(x) = x', g(y) = y',$$

the map is well defined as a different choice of inverse images x and y leads to a difference that belongs to T. This map in turn defines a linear map $\beta : M' \otimes N' \to (M \otimes N)/T$, which is clearly an inverse of α.

We may reformulate this as (writing 1 for the identity maps):

Suppose given exact sequences

$$0 \longrightarrow K \xrightarrow{\;i\;} M \xrightarrow{\;f\;} M' \longrightarrow 0$$

$$0 \longrightarrow L \xrightarrow{\;j\;} N \xrightarrow{\;g\;} N' \longrightarrow 0$$

then $M' \otimes N' \cong (M \otimes N)/T$, where

$$T = (i \otimes 1)(K \otimes N) + (1 \otimes j)(M \otimes L).$$

It now follows that if

$$M_1 \xrightarrow{\;f\;} M_2 \xrightarrow{\;g\;} M_3 \longrightarrow 0$$

is an exact sequence then so is

$$M_1 \otimes N \xrightarrow{f \otimes 1} M_2 \otimes N \xrightarrow{g \otimes 1} M_3 \otimes N \longrightarrow 0.$$

Change of coefficient ring

Let A and B be rings (commutative with identity), and P a two-sided $A - B$ module; that is for $a \in A$, $b \in b$ and $x \in P$ the products ax and xb are defined, and in addition to the usual conditions for A-modules and B-modules we assume that $(ax)b = a(xb)$. Then multiplication by an element $b \in B$ induces an A-linear map of P to itself, which we continue to denote by b. This determines a map $1 \otimes b : M \otimes_A P \to M \otimes_A P$ for any A-module M, and by definition we take this to be scalar multiplication by b in $M \otimes_A P$; that is, we set $(\sum y_i \otimes x_i)b = \sum y_i \otimes x_i b$ for $y_i \in M$ and $x_i \in P$.

If N is a B-module, then for $\varphi \in Hom_B(P, N)$ we define the product φa of φ and $a \in A$ by

$$(\varphi a)(x) = \varphi(ax) \qquad \text{for } x \in P;$$

we have $\varphi a \in Hom_B(P, N)$, and this makes $Hom_B(P, N)$ into an A-module. It is easy to show the following:

$$Hom_A(M, Hom_B(P, N)) \cong Hom_B(M \otimes_A P, N). \tag{7}$$

$$(M \otimes_A P) \otimes_B N \cong M \otimes_A (P \otimes_B N). \tag{8}$$

Given a ring homomorphism $\lambda : A \to B$, we can think of B as a two-sided $A - B$ module by setting $ab = \lambda(a)b$; then for any A-module M, $M \otimes_A B$ is a B-module, called the *extension of scalars* in M from A to B, and written $M_{(B)}$. For A-modules M and M' the following formula holds, so that tensor product commutes with change of scalars.

$$(M \otimes_A B) \otimes_B (M' \otimes_A B) = (M \otimes_A M') \otimes_A B. \tag{9}$$

Tensor product of A-algebras

We will assume that all ring homomorphisms take unit elements to unit elements. Given a ring homomorphism $\lambda : A \to B$ we say that B is an A-algebra. Let B' be another A-algebra defined by $\lambda' : A \to B'$. We say that a map $f : B \to B'$ is a homomorphism of A-algebras if it is a ring homomorphism satisfying $\lambda' = f \circ \lambda$. If B and C are A-algebras, then we can take the tensor product $B \otimes_A C$ of B and C as A-modules and this is again an A-algebra, with product given by

$$\left(\sum_i b_i \otimes c_i\right)\left(\sum_j b'_j \otimes c'_j\right) = \sum_{i,j} b_i b'_j \otimes c_i c'_j,$$

and the ring homomorphism $A \to B \otimes C$ given by $a \mapsto a \otimes 1 = 1 \otimes a$. The fact that the above product is well-defined can be seen by using the bilinearity of $bb' \otimes cc'$ with respect to both (b, c) and (b', c'). The algebra $B \otimes C$ contains $B \otimes 1$ and $1 \otimes C$ as subalgebras and is generated by these. Note that $B \otimes 1$ is not necessarily isomorphic to B.

References

[**Ma**] Hideyuki Matsumura, *Commutative Ring Theory*, Cambridge studies in advanced mathematics **8**, Cambridge University Press.