Trace in categories

§1 We first recall some linear algebra. Fix a field $\mathbb{C}$ and let V be a finite dimensional vector space. Let $V^* = \mathrm{Hom}(V, \mathbb{C})$ be the dual vector space. Then we have linear maps:

$$\begin{aligned} \varepsilon_V : V^* \otimes V &\longrightarrow \mathbb{C}, \\ f \otimes v &\longmapsto f(v); \\ \eta_V : \mathbb{C} &\longrightarrow V \otimes V^* \\ 1 &\longmapsto \sum_i v_i \otimes v^i ; \end{aligned}$$

where $\{v_i\}$ and $\{v^i\}$ are dual bases in V and V^*, respectively.

Furthermore, the compositions

$$V = \mathbb{C} \otimes V \xrightarrow{\eta_V \otimes id_V} V \otimes V^* \otimes V \xrightarrow{id_V \otimes \varepsilon_V} V \otimes \mathbb{C} = V$$

and

$$V^* = V^* \otimes \mathbb{C} \xrightarrow{id_{V^*} \otimes \eta_V} V^* \otimes V \otimes V^* \xrightarrow{\varepsilon_V \otimes id_{V^*}} \mathbb{C} \otimes V^* = V^*$$

are equal to the identity maps id_V and id_{V^*}, respectively.

We also have a map

$$\begin{aligned} \mathfrak{z}_V : V &\longrightarrow V^{**} \\ v &\longmapsto (f \longmapsto f(v)), \quad f \in V^* \}. \end{aligned}$$

The trace of an endomorphism $\varphi \in \mathrm{End}(V)$ is defined as the composition

$$\mathbb{C} \xrightarrow{\eta_V} V \otimes V^* \xrightarrow{\varphi \otimes \mathrm{id}_{V^*}} V \otimes V^* \xrightarrow{\mathfrak{z}_V \otimes \mathrm{id}_{V^*}} V^{**} \otimes V^* \xrightarrow{\varepsilon_{V^*}} \mathbb{C}.$$

§2 Everything in this section should be compared with §1.

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. Let (F^*, F) be an adjoint pair of functors, $F : \mathcal{C} \to \mathcal{D}$ and $F^* : \mathcal{D} \to \mathcal{C}$. These are the data of two natural transformations

$$\varepsilon_F : F^*F \longrightarrow \mathrm{id}_{\mathcal{C}}, \quad (\text{the counit});$$
$$\eta_F : \mathrm{id}_{\mathcal{D}} \longrightarrow FF^*, \quad (\text{the } \cancel{\text{co}}\text{unit});$$

such that the compositions:

$$F \xrightarrow{\eta_F \circ 1\!\!1_F} FF^*F \xrightarrow{1\!\!1_F \circ \varepsilon_F} F$$

and

$$F^* \xrightarrow{1\!\!1_{F^*} \circ \eta_F} F^*FF^* \xrightarrow{\varepsilon_F \circ 1\!\!1_{F^*}} F^*$$

are equal to the identity maps $1\!\!1_F : F \longrightarrow F$ and $1\!\!1_{F^*} : F^* \longrightarrow F^*$, respectively.

In addition, suppose that (F^{**}, F^*) is an adjoint pair for some functor $F^{**} : \mathcal{C} \longrightarrow \mathcal{D}$, and that we have a fixed natural transformation

$$\mathfrak{z}_F : F \longrightarrow F^{**}.$$

Then define a map $\mathrm{tr}: \mathrm{End}(F) \longrightarrow \mathrm{End}(\mathrm{id}_D)$ by assigning $\varphi \in \mathrm{End}(F)$ to the composition

$$\mathrm{id}_D \xrightarrow{\eta_F} FF^* \xrightarrow{\varphi \circ \mathbb{1}_{F^*}} FF^* \xrightarrow{z_F \circ \mathbb{1}_{F^*}} F^{**}F^* \xrightarrow{\varepsilon_{F^*}} \mathrm{id}_D .$$

Further, suppose that in the situation above $D = e$. Then for each $k \in \mathbb{Z}_{>0}$ we have a map

$$\mathrm{tr}: \mathrm{End}(F^k) \longrightarrow \mathrm{End}(F^{k-1}).$$

Iterating these maps we obtain a map

$$\mathrm{End}(F^k) \longrightarrow \mathrm{End}(\mathrm{id}_e)$$

called the <u>Markov trace</u>.