SUMMARY OF SOME CONSTRUCTIONS IN DERIVED CATEGORIES

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1. Review of categories and functors

Let $F, G : \mathcal{A} \to \mathcal{B}$ be functors between categories $\mathcal{A}$ and $\mathcal{B}$. A morphism of functors $\phi\colon F \to G$ consists of a morphism $\phi_X\colon F(X) \to G(X)$ for each $X \in \mathcal{A}$, such that $\phi_Y \circ F(f) = G(f) \circ \phi_X$ for every morphism $f\colon X \to Y$. The terms 'functorial', 'natural' and 'canonical' will be used as synonyms for 'a morphism of functors'. The identity endomorphism of a functor F will be denoted $\mathbb{1}_F$.

1.1. A functor $F\colon \mathcal{A} \to \mathcal{B}$ is *full* if the map it induces on Hom sets is surjective; it is *faithful* if the induced map is injective. It is an *equivalence* if there exists a functor $G : \mathcal{B} \to \mathcal{A}$ such that FG and GF are canonically isomorphic to $\mathrm{id}_{\mathcal{B}}$ and $\mathrm{id}_{\mathcal{A}}$, respectively. In this situation the functors F and G are *mutually inverse equivalences*. An equivalence is necessarily full and faithful. Moreover:

Proposition. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a full and faithful functor. Then F is an equivalence if and only if every object $Y \in \mathcal{B}$ is isomorphic to $F(X)$ for some $X \in \mathcal{A}$.*

Proof. See [KaSc, Prop. 1.3.13]. □

1.2. **Yoneda lemma.** Let $\mathcal{A}$ be a category. Let $\mathcal{S}\mathrm{et}$ be the category of sets. Let $\mathrm{Funct}(\mathcal{A}, \mathcal{S}\mathrm{et})$ be the category of functors $\mathcal{A} \to \mathcal{S}\mathrm{et}$. A functor $F \in \mathrm{Funct}(\mathcal{A}, \mathcal{S}\mathrm{et})$ is *representable* if $F \simeq \mathrm{Hom}_{\mathcal{A}}(X, -)$ for some object $X \in \mathcal{A}$. In this situation, the object X is said to *represent* F.

Lemma (Yoneda lemma)**.** *The functor*

$$\mathcal{A} \to \mathrm{Funct}(\mathcal{A}, \mathcal{S}\mathrm{et}), \qquad X \mapsto \mathrm{Hom}_{\mathcal{A}}(X, -)$$

defines an equivalence of $\mathcal{A}$ with the full subcategory of representable functors in $\mathrm{Funct}(\mathcal{A}, \mathcal{S}\mathrm{et})$.

Proof. See [KaSc, Prop. 1.4.3]. □

1.3. **Additive categories.** A category $\mathcal{A}$ is *additive* if all Hom sets are equipped with an abelian group structure such that composition of morphisms is bilinear and if all finite products exist in $\mathcal{A}$. The empty product gives a terminal object in $\mathcal{A}$. For $X, Y \in \mathcal{A}$, the maps $X \xleftarrow{\mathrm{id}} X \xrightarrow{0} Y$ give a unique map $X \to X \times Y$. Similarly, there is a unique map $Y \to X \times Y$. Consequently, finite products coincide with the corresponding coproducts. In particular, the terminal object is also initial and is hence a zero object.

Let $\mathcal{B}$ be another additive category. An *additive functor* $\mathcal{A} \to \mathcal{B}$ is a functor F such that $F(f + g) = F(f) + F(g)$ for all morphisms $f, g \in \mathcal{A}$. Functors between additive categories will always be assumed to be additive.

1.4. **Abelian categories.** An additive category is *abelian* if it possesses all kernels, cokernels and if every monomorphism is the kernel of some morphism and every epimorphism is the cokernel of some morphism. See [KaSc, Ch. 8] for details. Let $\mathcal{A}$ be an abelian category. A sequence of maps $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_n} X_{n+1}$, in $\mathcal{A}$, is an *exact sequence* if the image of f_i is equal to the kernel of f_{i+1} for each $0 \leq i < n$. An exact sequence $0 \to X \to Y \to Z \to 0$ is also referred to as a *short exact sequence.*

Let $\mathcal{B}$ be another abelian category. A functor $F\colon \mathcal{A} \to \mathcal{B}$ is *left exact* if for each exact sequence $0 \to X \to Y$ in $\mathcal{A}$, the sequence $0 \to F(X) \to F(Y)$ is exact in $\mathcal{B}$. Similarly, F is *right exact* if for each exact sequence $X \to Y \to 0$ in $\mathcal{A}$, the sequence $F(X) \to F(Y) \to 0$ is exact in $\mathcal{B}$. The functor F is *exact* if it is both left and right exact.

The *Grothendieck group* $K_0(\mathcal{A})$ is the free abelian group on symbols $[X]$, $X \in \mathcal{A}$, modulo the relation $[X] = [X_1] + [X_2]$ for each short exact sequence $0 \to X_1 \to X \to X_2 \to 0$. Consequently, if $X^\bullet = \cdots \to X^i \to \cdots$ is a bounded complex in $\mathcal{A}$, then $\sum_i (-1)^i [X^i] = \sum_i (-1)^i [H^i(X^\bullet)]$ in $K_0(\mathcal{A})$.

Let $\{L_i\}$ be a set of objects in $\mathcal{A}$ such that the classes $[L_i]$ comprise a basis of $K_0(\mathcal{A})$. Then for $M \in \mathcal{A}$, we write $[M : L_i]$ for the coefficient of L_i when $[M]$ is expanded in terms of the basis $\{[L_i]\}$, i.e., $[M] = \sum_i [M : L_i][L_i]$.

A *simple object* or an *object of length one* is an object $L \in \mathcal{A}$ such that any monomorphism $A \to L$ is either 0 or an isomorphism. For $n \geq 2$, *objects of length n* are inductively defined to be those objects X that fit into an exact sequence $0 \to X' \to X \to L \to 0$, with X' of length $n-1$ and L simple. If every object in $\mathcal{A}$ has finite length, then the Jordan-Hölder theorem holds in $\mathcal{A}$, i.e., for an object $X \in \mathcal{A}$, the length of X is well defined and the simple objects that occur in a 'composition series' of X are unique up to isomorphism and permutation (see [KaSc, Exer. 8.20]).

1.5. **Complexes.** Let $\mathcal{A}$ be an additive category. A *complex* $X^\bullet$ in $\mathcal{A}$ is the data of a $\mathbb{Z}$-graded object $X^\bullet = \bigoplus_{i\in\mathbb{Z}} X^i$, $X^i \in \mathcal{A}$ and a degree 1 endomorphism $d_X : X^\bullet \to X^\bullet$ such that $d_X^2 = 0$. This is usually visualized as a sequence of morphisms $\cdots \to X^i \xrightarrow{d_i} X^{i+1} \to \cdots$, such that $d_{i+1} \circ d_i = 0$ for each i. The object X^i is in degree i and the morphisms d_i are those induced by d_X. The endomorphism d_X is the *differential* of $X^\bullet$. If $\mathcal{A}$ is an abelian category, the cohomology $H^*(X^\bullet)$ of $X^\bullet$ is the sequence of objects (in $\mathcal{A}$): $H^i(X^\bullet) = \frac{\ker(d_i)}{\operatorname{im}(d_{i-1})}$.

A *chain map* is a graded morphism $f\colon X^\bullet \to Y^\bullet$ of degree 0 such that $d_Y f = f d_X$. Let $f, g : X^\bullet \to Y^\bullet$ be chain maps. Then f and g are *homotopic* if there exists a graded morphism $s : X^\bullet \to Y^\bullet$ of degree -1 such that $d_Y s + s d_X = f - g$. The map s is a *homotopy* between f and g. Further, we say that f and g are in the same *homotopy class.* In the setting of abelian categories, homotopic maps induce the same maps on cohomology (see [KaSc, Lemma 12.2.2]).

Denote by $\mathrm{Comp}(\mathcal{A})$ the category of all complexes, by $\mathrm{Comp}^-(\mathcal{A})$ the category of bounded above complexes, by $\mathrm{Comp}^+(\mathcal{A})$ the category of bounded below complexes and by $\mathrm{Comp}^{\mathrm{b}}(\mathcal{A})$ the category of bounded complexes, in $\mathcal{A}$. As each object of $\mathcal{A}$ is a complex concentrated in degree 0 we obtain a full and faithful embedding $\mathcal{A} \hookrightarrow \mathrm{Comp}(\mathcal{A})$.

The *shift functor* $[1]\colon \mathrm{Comp}(\mathcal{A}) \to \mathrm{Comp}(\mathcal{A})$ is defined as follows: if $X^\bullet$ is a complex with differential d_i, then $(X^\bullet[1])^i = X^{i+1}$ with differential $d_i' = -d_{i+1}$. It is clear that $[1]$ is a self-equivalence of $\mathrm{Comp}(\mathcal{A})$. For $n \in \mathbb{Z}$, set $[n] = [1]^n$.

Let $X^\bullet, Y^\bullet$ be complexes in $\mathcal{A}$ with differentials d'_i and d''_i, respectively. Let $\phi\colon X^\bullet \to Y^\bullet$ be a chain map. The *cone* of ϕ is

$$\operatorname{cone}(\phi)^i = Y^i \oplus X^{i+1} \quad \text{with differential} \quad d_i = \begin{pmatrix} d''_i & \phi_{i+1} \\ 0 & -d'_{i+1} \end{pmatrix}. \tag{1.1}$$

Define $\iota\colon Y^\bullet \to \operatorname{cone}(\phi)$ by $Y^i \xrightarrow{\binom{\mathrm{id}}{0}} Y^i \oplus X^{i+1}$ and $\delta\colon \operatorname{cone}(\phi) \to X^\bullet[1]$ by $Y^i \oplus X^{i+1} \xrightarrow{(0\ \mathrm{id})} X^{i+1}$. Both inc and δ are chain maps. A *standard triangle* is a sequence of morphisms of the form

$$X^\bullet \xrightarrow{\phi} Y^\bullet \xrightarrow{\iota} \operatorname{cone}(\phi) \xrightarrow{\delta} X^\bullet[1]. \tag{1.2}$$

2. Triangulated categories

A triangulated category is an additive category endowed with an auto-equivalence and a family of so-called *distinguished triangles* satisfying certain axioms. This subject deserves a whole book such as [Neeman]. I will not try to give an introduction to triangulated categories. However, I only assume that the reader is familiar with the axiomatics and basic properties of a triangulated category at the level of [KaSc, Ch. 10 §1]. The purpose of the remainder of this note is to recall a few specific constructions.

2.1. The *shift functor* in a triangulated category will be denoted by [1]. For $n \in \mathbb{Z}$, set $[n] = [1]^n$. Let $\mathcal{T}$ be a triangulated category. A distinguished triangle $X \to Y \to Z \to X[1]$ will often be written as $X \to Y \to Z \rightsquigarrow$. Further, Z will be referred to as the *cone* of the map $X \to Y$. Similarly, X will be referred to as the *cocone* of the map $Y \to Z$.

2.2. The *Grothendieck group* $K_0(\mathcal{T})$ is the free abelian group on symbols $[X]$, $X \in \mathcal{T}$, modulo the relation $[X] = [X_1] + [X_2]$ for each distinguished triangle $X_1 \to X \to X_2 \rightsquigarrow$. In particular, $[X[1]] = -[X]$ (see [KaSc, §10.1, TR3]).

2.3. Let $\mathcal{T}'$ be another triangulated category. A *triangulated* or an *exact* functor $\mathcal{T} \to \mathcal{T}'$ is the data of a functor F that preserves distinguished triangles and a canonical isomorphism $F \circ [1] \xrightarrow{\sim} [1] \circ F$. A morphism $F \to F'$ between triangulated functors is a natural transformation θ such that the following diagram commutes

$$\begin{array}{ccc} F \circ [1] & \xrightarrow{\theta \circ [1]} & F' \circ [1] \\ \sim \downarrow & & \downarrow \sim \\ [1] \circ F & \xrightarrow{[1] \circ \theta} & [1] \circ F' \end{array}$$

All natural transformations between triangulated functors will tacitly be assumed to be morphisms of triangulated functors.

2.4. **Cohomological functors.** Let $\mathcal{A}$ be an abelian category. A functor $H\colon \mathcal{T} \to \mathcal{A}$ is *cohomological* if, for every distinguished triangle $X \to Y \to Z \rightsquigarrow$ in $\mathcal{T}$, the sequence $H(X) \to H(Y) \to H(Z)$ is exact in $\mathcal{A}$.

Proposition. *Let $\mathcal{T}$ be a triangulated category and let $X \in \mathcal{T}$. The functors* $\operatorname{Hom}_{\mathcal{T}}(X, -)$ *and* $\operatorname{Hom}_{\mathcal{T}}(-, X)$ *are cohomological.*

Proof. See [KaSc, Prop. 10.1.13]. □

2.5. Let $\mathcal{T}$ be a triangulated category. Let $\mathcal{A}, \mathcal{B} \subset \mathcal{T}$ be subcategories of $\mathcal{T}$. For $X \in \mathcal{T}$, write $[X] \in \mathcal{A}$ if there exists an object in $\mathcal{A}$ that is isomorphic to X. Set

$$\mathcal{A} * \mathcal{B} = \{Y \in \mathcal{T} \,|\, \text{there is a distinguished triangle } X \to Y \to Z \rightsquigarrow, \\ \text{with } [X] \in \mathcal{A} \text{ and } [Z] \in \mathcal{B}\}.$$

Lemma ([BBD, Lemme 1.3.10])**.** *The operation $*$ is associative. That is, if $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are subcategories of $\mathcal{T}$, then $(\mathcal{A} * \mathcal{B}) * \mathcal{C} = \mathcal{A} * (\mathcal{B} * \mathcal{C})$.*

Proof. Suppose $[X] \in (\mathcal{A} * \mathcal{B}) * \mathcal{C}$. Then there is some $X' \in \mathcal{T}$ and distinguished triangles $A \to X' \to B \rightsquigarrow$ and $X' \to X \to C \rightsquigarrow$, with $[A] \in \mathcal{A}$, $[B] \in \mathcal{B}$ and $[C] \in \mathcal{C}$. Apply the octahedron axiom (see [KaSc, Def. 10.1.6 TR5]) to the composition $A \to X' \to X$ to obtain distinguished triangles $A \to X \to BC \rightsquigarrow$ and $B \to X'' \to C \rightsquigarrow$, with $X'' \in \mathcal{T}$. Thus, $[X] \in \mathcal{A} * (\mathcal{B} * \mathcal{C})$. The reverse inclusion is proved similarly. ☐

Let $\mathcal{A} \subseteq \mathcal{T}$ be a subcategory. Inductively define $\mathcal{A}^{*i}$, $i \in \mathbb{Z}_{\geq 0}$, by $\mathcal{A}^{*0} = 0$ and $\mathcal{A}^{*i+1} = \mathcal{A} * \mathcal{A}^{*i}$. As $*$ is associative, $\mathcal{A}^{*i+1} = \mathcal{A} * \mathcal{A}^{*i} = \mathcal{A}^{*i} * \mathcal{A}$. Further, $\mathcal{A}^{*i} \subseteq \mathcal{A}^{*i+1}$. Set $\mathcal{A}^{*\infty} = \bigcup_{i \in \mathbb{Z}_{\geq 0}} \mathcal{A}^{*i}$.

2.6. **Filtrations.** An object $X \in \mathcal{T}$ is *filtered* by objects $Y_1, \ldots, Y_n$ if there exists a sequence of objects $0 = X_0, X_1, \ldots, X_n = X$ and distinguished triangles $X_{i-1} \to X_i \to Y_i \rightsquigarrow$.

Lemma. *Let $\mathcal{A} \subset \mathcal{T}$ be a subcategory. Then $X \in \mathcal{T}$ is in $\mathcal{A}^{*n}$ if and only if X is filtered by some $Y_1, \ldots, Y_n \in \mathcal{A}$.*

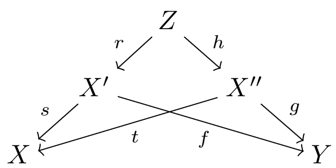
Proof. This is clear if one proceeds by induction on n. ☐

Remark. Filtrations in triangulated categories are most commonly used in the following situation: let H be a cohomological functor. Let X be filtered by $Y_1, \ldots, Y_n$. By definition, there is a sequence of objects $0 = X_0, \ldots, X_n = X$ and distinguished triangles $X_{i-1} \to X_i \to Y_i \rightsquigarrow$. Assume that $H(Y_i[m]) = 0$ for all $m \in \mathbb{Z}$ and $1 \leq i \leq n$. Then, proceeding by induction on n, it follows that $H(X[m]) = 0$ for all $m \in \mathbb{Z}$.

2.7. **Localization.** Let $\mathcal{T}$ be a triangulated category. Let $\mathcal{N} \subset \mathcal{T}$ be a *localizing* subcategory, i.e., $\mathcal{N}$ satisfies the following properties:

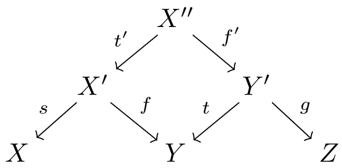
- $0 \in \mathcal{N}$;
- $N \in \mathcal{N}$ if and only if $N[1] \in \mathcal{N}$;
- if $N \to M \to N' \rightsquigarrow$ is a distinguished triangle in $\mathcal{T}$ with $N, N' \in \mathcal{N}$, then $M \in \mathcal{N}$.

An $\mathcal{N}$-*quasi-isomorphism*, or simply *quasi-isomorphism* if the $\mathcal{N}$ is clear, is a morphism $s\colon X \to Y$ in $\mathcal{T}$ such that there is a distinguished triangle $X \xrightarrow{s} Y \to Z \rightsquigarrow$ with $Z \in \mathcal{N}$. Let $\mathcal{N}-$qis denote the collection of $\mathcal{N}$-quasi-isomorphisms. A *roof* (s, f) is a diagram of the form $X \xleftarrow{s} X' \xrightarrow{f} Y$, with $s \in \mathcal{N}-$qis. Define an equivalence relation on roofs by declaring $X \xleftarrow{s} X' \xrightarrow{f} Y$ and $X \xleftarrow{t} X'' \xleftarrow{g} Y$ to be equivalent if there exists a third roof $X' \xleftarrow{r} Z \xrightarrow{h} X''$ such that the following diagram commutes



This equivalence relation is reflexive, symmetric and transitive (see [GeMa, Ch. 3 §2, Lemma 8 (a)] or [KaSc, Lemma 7.1.12])

Given roofs $X \overset{s}{\leftarrow} X' \overset{f}{\rightarrow} Y$ and $Y \overset{t}{\leftarrow} Y' \overset{g}{\rightarrow} Z$, there is a roof $X' \overset{t'}{\longleftarrow} X'' \overset{f'}{\longrightarrow} Y'$ such that the following diagram commutes



The roof $X \overset{st'}{\longleftarrow} X'' \overset{gf'}{\longrightarrow} Z$ is defined to be the composition of $X \overset{s}{\leftarrow} X' \overset{f}{\rightarrow} Y$ and $Y \overset{t}{\leftarrow} Y' \overset{g}{\rightarrow} Z$. This operation is well defined and associative on equivalence classes of roofs. For details see [GeMa, Ch. 3 §2 Lemma 8 (b)] or [KaSc, Lemma 7.1.13].

The *localization* of $\mathcal{T}$ with respect to $\mathcal{N}$, denoted $\mathcal{T}/\mathcal{N}$, is the following category:

- $\mathrm{Objects}(\mathcal{T}/\mathcal{N}) = \mathrm{Objects}(\mathcal{T})$;
- $\mathrm{Hom}_{\mathcal{T}/\mathcal{N}}(X, Y) =$ equivalence classes of roofs $X \overset{s}{\leftarrow} X' \overset{f}{\rightarrow} Y$,

with composition of roofs defined as above.

The localization functor $\mathrm{quot}\colon \mathcal{T} \to \mathcal{T}/\mathcal{N}$ is defined to be the identity on objects and by sending $f\colon X \to Y$ in $\mathcal{T}$ to the roof $X \overset{\mathrm{id}}{\longleftarrow} X \overset{f}{\rightarrow} Y$. We abuse notation and write $[1]\colon \mathcal{T}/\mathcal{N} \to \mathcal{T}/\mathcal{N}$ for the image of $[1]\colon \mathcal{T} \to \mathcal{T}$ under quot.

Proposition. *Define distinguished triangles in* $\mathcal{T}/\mathcal{N}$ *as sequences equivalent to the image (under* quot*) of a distinguished triangle in* $\mathcal{T}$.

(i) $\mathcal{T}/\mathcal{N}$ *is a triangulated category and* $\mathrm{quot} : \mathcal{T} \to \mathcal{T}/\mathcal{N}$ *is a triangulated functor.*
(ii) *If* $N \in \mathcal{N}$, *then* $\mathrm{quot}(N) = 0$.
(iii) *Let* $\mathcal{T}'$ *be a triangulated category and let* $\mathcal{F} : \mathcal{T} \to \mathcal{T}'$ *be a triangulated functor such that* $F(N) = 0$ *for each* $N \in \mathcal{N}$. *Then* F *factors uniquely through* quot.

Proof. See [KaSc, Thm. 10.2.3]. □

2.8. **The homotopy category.** Let $\mathcal{A}$ be an additive category. The *homotopy category* of $\mathcal{A}$, denoted $\mathrm{Ho}(\mathcal{A})$, is defined as follows:

- $\mathrm{Objects}(\mathrm{Ho}(\mathcal{A})) = \mathrm{Objects}(\mathrm{Comp}(\mathcal{A}))$;
- $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{A})}(X^\bullet, Y^\bullet) =$ homotopy classes of maps in $\mathrm{Hom}_{\mathrm{Comp}(\mathcal{A})}(X^\bullet, Y^\bullet)$.

Replacing $\mathrm{Comp}(\mathcal{A})$ by $\mathrm{Comp}^+(\mathcal{A})$, $\mathrm{Comp}^-(\mathcal{A})$ or $\mathrm{Comp}^\mathrm{b}(\mathcal{A})$ in the definition above we obtain the variants $\mathrm{Ho}^+(\mathcal{A})$, $\mathrm{Ho}^-(\mathcal{A})$ and $\mathrm{Ho}^\mathrm{b}(\mathcal{A})$, respectively.

Theorem. *Let* $[1]\colon \mathrm{Ho}(\mathcal{A}) \to \mathrm{Ho}(\mathcal{A})$ *be the shift functor on complexes. Define distinguished triangles in* $\mathrm{Ho}(\mathcal{A})$ *to be triangles isomorphic to* (1.2). *This endows* $\mathrm{Ho}(\mathcal{A})$ *with the structure of a triangulated category.*

Proof. See [KaSc, Thm. 11.2.6]. □

Proposition. *Let* $\mathcal{A}$ *be an abelian category For* $n \in \mathbb{Z}$, *let* $H^n\colon \mathrm{Ho}(\mathcal{A}) \to \mathcal{A}$ *be the functor that associates to a complex its* n^{th} *cohomology. Then* H^n *is cohomological.*

Proof. See [KaSc, Cor. 12.2.5]. □

2.9. **The derived category.** Let $\mathcal{A}$ be an abelian category. Let $\mathcal{N} \subset \mathrm{Ho}(\mathcal{A})$ be the subcategory consisting of complexes $X^\bullet$ such that $H^i(X^\bullet) = 0$ for all $i \in \mathbb{Z}$. Then $\mathcal{N}$ is a localizing subcategory (for details see [KaSc, Ch. 13 §1]). So we are in the setting of Prop. **2.7**. The *derived category* of $\mathcal{A}$, denoted $\mathrm{D}(\mathcal{A})$, is the triangulated category $\mathrm{Ho}(\mathcal{A})/\mathcal{N}$. Replacing $\mathrm{Ho}(\mathcal{A})$ by $\mathrm{Ho}^+(\mathcal{A})$, $\mathrm{Ho}^-(\mathcal{A})$ or $\mathrm{Ho}^\mathrm{b}(\mathcal{A})$ in this definition, we obtain the variants $\mathrm{D}^+(\mathcal{A})$, $\mathrm{D}^-(\mathcal{A})$ and $\mathrm{D}^\mathrm{b}(\mathcal{A})$, respectively. The category $\mathrm{D}^\mathrm{b}(\mathcal{A})$

(resp. $\mathrm{D}^+(\mathcal{A})$, resp. $\mathrm{D}^-(\mathcal{A})$) is equivalent to the full subcategory of $\mathrm{D}(\mathcal{A})$ consisting of complexes $X^\bullet$ such that $H^n(X^\bullet) = 0$ for $|n| >> 0$ (resp. $n << 0$, resp. $n >> 0$), see [KaSc, Prop. 13.1.12] for details.

Proposition. *For $n \in \mathbb{Z}$, let $H^n \colon \mathrm{D}(\mathcal{A}) \to \mathcal{A}$ be the functor that sends a complex to its n^{th} cohomology. Then H^n is cohomological.*

Proof. See [KaSc, Prop. 13.1.5]. □

2.10. **Exact sequences and distinguished triangles.**

Proposition. *Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be an exact sequence in $\mathrm{Comp}(\mathcal{A})$. Then there exists a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \rightsquigarrow$ in $\mathrm{D}(\mathcal{A})$.*

Proof. See [KaSc, Prop. 13.1.13] □

2.11. **Hom in the derived category.** Let $X, Y \in \mathrm{D}(\mathcal{A})$. Set $\mathrm{Ext}^k_{\mathcal{A}}(X, Y) = \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(X, Y[k])$. An object $X \in \mathcal{A}$ is also an object of $\mathrm{D}(\mathcal{A})$, since X is a complex concentrated in degree 0.

Proposition. *Let $X, Y \in \mathcal{A}$. Then*

(i) $\mathrm{Ext}^k_{\mathcal{A}}(X, Y) = 0$ *for $k < 0$;*
(ii) $\mathrm{Ext}^0_{\mathcal{A}}(X, Y) \simeq \mathrm{Hom}_{\mathcal{A}}(X, Y)$. *That is, the natural functor $\mathcal{A} \to \mathrm{D}(\mathcal{A})$ is full and faithful.*

Proof. See [KaSc, Prop. 13.1.10]. □

2.12. **Relation between Grothendieck groups.** The embedding $\mathcal{A} \to \mathrm{D}^{\mathrm{b}}(\mathcal{A})$ induces a map $K_0(\mathcal{A}) \to K_0(\mathrm{D}^{\mathrm{b}}(\mathcal{A})$. This map is an isomorphism, the inverse is given by $[X^\bullet] \mapsto \sum_{i \in \mathbb{Z}} (-1)^i [H^i(X^\bullet)]$. The groups $K_0(\mathcal{A})$ and $K_0(\mathrm{D}^{\mathrm{b}}(\mathcal{A}))$ are identified via this isomorphism.

2.13. **Yoneda Ext.** Let $X, Y \in \mathcal{A}$. Let $Z = 0 \to Y \to Z_1 \to \cdots \to Z_n \to X \to 0$ be an exact sequence in $\mathcal{A}$. Define $\theta(Z) \in \mathrm{Ext}^n_{\mathcal{A}}(X, Y)$ by the roof

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & X & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow \\
0 & \longrightarrow & Y & \longrightarrow & Z_1 & \longrightarrow & \cdots & \longrightarrow & Z_n & \longrightarrow & 0 \\
\downarrow & & \| & & \downarrow & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & Y & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

(The top vertical arrow is a quasi-isomorphism).

Proposition. *Each element of $\mathrm{Ext}^n_{\mathcal{A}}(X, Y)$ is of the form $\theta(Z)$ for some exact sequence $Z = 0 \to Y \to Z_1 \to \cdots \to Z_n \to X \to 0$ in $\mathcal{A}$. Further:*

(i) *$\theta(Z) = 0$ if and only if there exists a commutative diagram with exact rows*

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & Z_1 & \longrightarrow & Z_2' & \longrightarrow & \cdots & \longrightarrow & Z_n & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & Z_1 & \longrightarrow & Z_2 & \longrightarrow & \cdots & \longrightarrow & Z_n & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

(ii) *If $Z' = 0 \to Y \to Z_1' \to \cdots \to Z_n' \to X \to 0$ is another exact sequence in $\mathcal{A}$, then $\theta(Z) = \theta(Z')$ if and only if there exists a commutative diagram*

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & Y & \longrightarrow & Z_1 & \longrightarrow & \cdots & \longrightarrow & Z_n & \longrightarrow & X & \longrightarrow & 0 \\
 & & \| & & \downarrow & & & & \downarrow & & \| & & \\
0 & \longrightarrow & Y & \longrightarrow & Z_1' & \longrightarrow & \cdots & \longrightarrow & Z_n' & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

Proof. For the first statement and (i), see [KaSc, Exer. 13.16] or [GeMa, Ch. III §5, Thm. 5 (c)]. (ii) is a restatement of the equivalence relation on roofs (see §**2.7**). □

Corollary. *Let* $Z = 0 \to Y \to Z_1 \to X \to 0$ *be a short exact sequence. Then* $\theta(Z) \in \mathrm{Ext}^1_{\mathcal{A}}(X, Y)$ *is zero if and only if* Z *is split exact.*

2.14. **Cup product.** Let $Z' = 0 \to Y' \to Z_1 \to \cdots \to Z_m \to Y \to 0$ and $Z = 0 \to Y \to Z_{m+1} \to \cdots \to Z_{m+n} \to X \to 0$ be exact sequences in $\mathcal{A}$. Let $Z' \cup Z$ denote the exact sequence $0 \to Z_1 \to \cdots \to Z_m \to Z_{m+1} \to \cdots \to Z_{m+n} \to X \to 0$.

Proposition. $\theta(Z' \cup Z) = \theta(Z') \circ \theta(Z)$.

Proof. See [GeMa, Ch. 3 §5, Thm. 5 (c)]. □

2.15. **Projectives and injectives.** Let $\mathcal{A}$ be an abelian category. An object $P \in \mathcal{A}$ is *projective* if $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is exact. The category $\mathcal{A}$ has *enough projectives* if for any $A \in \mathcal{A}$ there exists an epimorphism $P \twoheadrightarrow A$ with P projective. Let $P_L \twoheadrightarrow L$ be an epimorphism with P_L projective and $L \in \mathcal{A}$ simple. Then P_L is a *projective cover* of L if P_L is indecomposable (i.e., P_L cannot be written as a non-trivial direct sum). A projective cover is unique up to isomorphism.

An object $I \in \mathcal{A}$ is *injective* if $\mathrm{Hom}_{\mathcal{A}}(-, I)$ is exact. The category $\mathcal{A}$ has *enough injectives* if for any $A \in \mathcal{A}$ there exists a monomorphism $A \hookrightarrow I$ with I injective. Let $L \hookrightarrow I_L$ be a monomorphism with I_L injective and $L \in \mathcal{A}$ simple. Then I_L is an *injective hull* of L if I_L is indecomposable. An injective hull is unique up to isomorphism.

Proposition. *Let* $\mathcal{A}$ *be an abelian category. Let* $X \in \mathcal{A}$. *The following are equivalent:*

(i) X *is projective.*
(ii) $\mathrm{Ext}^1_{\mathcal{A}}(X, Y) = 0$ *for all* $Y \in \mathcal{A}$.
(iii) $\mathrm{Ext}^n_{\mathcal{A}}(X, Y) = 0$ *for all* $Y \in \mathcal{A}$ *and all* $n \neq 0$.

Similarly, the following are equivalent:

(i) X *is injective.*
(ii) $\mathrm{Ext}^1_{\mathcal{A}}(Y, X) = 0$ *for all* $Y \in \mathcal{A}$.
(iii) $\mathrm{Ext}^n_{\mathcal{A}}(Y, X) = 0$ *for all* $Y \in \mathcal{A}$ *and all* $n \neq 0$.

Proof. See [GeMa, Ch. III §5, Lemma 10]. □

Remark. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $f_* \colon \mathcal{A} \to \mathcal{B}$ be right adjoint to $f^* \colon \mathcal{B} \to \mathcal{A}$. Assume that f_* is exact. Let $P \in \mathcal{B}$ be projective. Then f^*P is projective in $\mathcal{A}$, since $\mathrm{Hom}_{\mathcal{A}}(f^*P, -) \simeq \mathrm{Hom}_{\mathcal{B}}(P, f_*-)$. A similar statement holds for injectives.

2.16. **Derived categories as homotopy categories.**

Proposition. *Let* $\mathcal{A}$ *be an abelian category. Let* $\mathcal{N} \subset \mathrm{Ho}(\mathcal{A})$ *be the subcategory consisting of complexes* $X^\bullet$ *such that* $H^i(X^\bullet) = 0$ *for all* $i \in \mathbb{Z}$. *Let* $\mathcal{I}$ *be a full subcategory of* $\mathcal{A}$ *such that for any* $X \in \mathcal{A}$, *there exists* $I \in \mathcal{I}$ *and a monomorphism* $X \hookrightarrow I$. *Then*

(i) *for any* $X \in \mathrm{Ho}^+(\mathcal{A})$, *there exists* $I \in \mathrm{Ho}^+(\mathcal{I})$ *and a quasi-isomorphism* $s \colon X \to I$;
(ii) *let* $\mathcal{N}' = \mathcal{N} \cap \mathrm{Ho}^+(\mathcal{I})$. *The obvious functor* $\mathrm{Ho}^+(\mathcal{I})/\mathcal{N}' \to \mathrm{D}^+(\mathcal{A})$ *is a triangulated equivalence of categories.*

Proof. (i) is [KaSc, Lemma 13.2.1], (ii) is [KaSc, Prop. 13.2.2 (i)]. □

Lemma. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{I} \subseteq \mathcal{A}$ be the full subcategory consisting of injective objects. Let $I^\bullet \in \mathrm{Comp}^+(\mathcal{I})$. Let $X^\bullet \in \mathrm{Comp}(\mathcal{A})$ be such that the cohomology of $X^\bullet$ is zero in every degree. Let $f\colon X^\bullet \to I^\bullet$ be a chain map. Then f is homotopic to zero.*

Proof. See [KaSc, Lemma 13.2.4]. □

Combining Prop. **2.16** and Lemma **2.16** we get:

Theorem. *Let $\mathcal{A}$ be an abelian category and let $\mathcal{I}$ be the full subcategory of $\mathcal{A}$ consisting of injective objects. If $\mathcal{A}$ has enough injectives, then $\mathrm{Ho}^+(\mathcal{I})$ is equivalent to $\mathrm{D}^+(\mathcal{A})$ as a triangulated category.*

Proof. See [KaSc, Prop. 13.2.3]. □

Assume we are in the situation of Prop. **2.16** (i), i.e., we are given a quasi-isomorphism $s\colon X \to I$ with $X \in \mathrm{Ho}^+(\mathcal{A})$ and $I \in \mathrm{Ho}^+(\mathcal{I})$, then I is a *resolution* of X by objects in $\mathcal{I}$.

2.17. **Derived functors.** Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and let $f_*\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor. A full additive subcategory $\mathcal{I}$ of $\mathcal{A}$ is *f_*-injective* if:

(i) for every object $X \in \mathcal{A}$ there is a monomorphism $X \hookrightarrow I$ with $I \in \mathcal{I}$;
(ii) if $0 \to X \to Y \to Z \to 0$ is an exact sequence in $\mathcal{A}$, and if X, Y are in $\mathcal{I}$, then Z is also in $\mathcal{I}$;
(iii) if $0 \to X \to Y \to Z \to 0$ is an exact sequence in $\mathcal{A}$ with $X, Y, Z \in \mathcal{I}$, then $0 \to f_*X \to f_*Y \to f_*Z \to 0$ is exact in $\mathcal{B}$.

If $\mathcal{A}$ has enough injectives, then the full subcategory of injective objects in $\mathcal{A}$ is f_*-injective for *any* left exact functor f_* (see [KaSc, Remark 13.3.6 (iii)]).

Let $\mathcal{N} \subset \mathrm{Ho}(\mathcal{A})$ be the subcategory consisting of complexes whose cohomology vanishes in every degree. Suppose an f_*-injective subcategory $\mathcal{I} \subseteq \mathcal{A}$ exists. Set $\mathcal{N}' = \mathcal{N} \cap \mathrm{Ho}^+(\mathcal{I})$. Since f_* preserves exact sequences consisting of objects in $\mathcal{I}$, it follows that f_* transforms objects of $\mathrm{Ho}^+(\mathcal{I})$ quasi-isomorphic to zero into objects of $\mathrm{Ho}^+(\mathcal{B})$ satisfying the same property. Therefore, $f_*\colon \mathrm{Ho}^+(\mathcal{I}) \to \mathrm{Ho}^+(\mathcal{B})$ factors through $\mathrm{Ho}^+(\mathcal{I})/\mathcal{N}'$. Let $\mathbf{i}\colon \mathrm{Ho}^+(\mathcal{I})/\mathcal{N}' \xrightarrow{\sim} D^+(\mathcal{A})$ be the equivalence inverse to the one described in Prop. **2.16** (i.e., if $X \in \mathrm{D}^+(\mathcal{A})$, then $\mathbf{i}X$ is a resolution of X by objects in $\mathcal{I}$). The *right derived functor* $\mathbf{R}f_*\colon \mathrm{D}^+(\mathcal{A}) \to \mathrm{D}^+(\mathcal{B})$ is defined to be the composition

$$\mathrm{D}^+(\mathcal{A}) \xrightarrow{\mathbf{i}} \mathrm{Ho}^+(\mathcal{I})/\mathcal{N}' \xrightarrow{f_*} \mathrm{Ho}^+(\mathcal{B}) \xrightarrow{\mathrm{quot}} \mathrm{D}^+(\mathcal{B}). \tag{2.1}$$

The derived functor $\mathbf{R}f_*$ is unique up to canonical isomorphism, in particular it does not depend on the choice of the f_*-injective subcategory $\mathcal{I}$ (see [KaSc, Prop. 13.3.5]).

Proposition. *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be abelian categories. Let $f_*\colon \mathcal{A} \to \mathcal{B}$ and $g_*\colon \mathcal{B} \to \mathcal{C}$ be left exact functors. Assume that there exist full additive subcategories $\mathcal{I} \subseteq \mathcal{A}$ and $\mathcal{I}' \subseteq \mathcal{B}$ such that $\mathcal{I}$ is f_*-injective, $\mathcal{I}'$ is g_*-injective and $f_*\mathcal{I} \subseteq \mathcal{I}'$. Then $\mathcal{I}$ is g_*f_*-injective and induces an isomorphism*

$$\mathbf{R}(g_*f_*) \xrightarrow{\sim} \mathbf{R}g_*\mathbf{R}f_*.$$

Proof. See [KaSc, Prop. 13.3.13 (ii)]. □

Remark. Similar statements apply to right exact functors. See [KaSc, Remark 13.3.14].

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