THE OCTAHEDRON AXIOM

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Let $\mathcal{T}$ be an additive category. The structure of a *triangulated category* on $\mathcal{T}$ is given by the following data:

(i) An additive equivalence $\Sigma : \mathcal{T} \xrightarrow{\sim} \mathcal{T}$. For $X \in \mathcal{T}$ and $n \in \mathbb{Z}$, we will write $X[n]$ instead of $\Sigma^n X$. Further, diagrams of the form

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

will be called *triangles*. A commutative diagram of the form

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow f[1] \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z & \xrightarrow{w'} & X'[1] \end{array}$$

will be called a *morphism of triangles*. If f, g, h are isomorphisms then we will say that the two triangles involved are isomorphic.

(ii) A class of *distinguished triangles* satisfying the following axioms:

TR1: For any $X \in \mathcal{T}$ the triangle

$$X \xrightarrow{\text{id}} X \longrightarrow 0 \longrightarrow X[1]$$

is distinguished and any triangle isomorphic to a distinguished one, is itself distinguished. Furthermore, any morphism $X \xrightarrow{u} Y$ can be be completed (not necessarily uniquely) to a distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1].$$

A distinguished triangle

$$X \to Y \to Z \to X[1]$$

will also be written as

$$X \to Y \to Z \rightsquigarrow$$

TR2: (Rotation invariance). A triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

is distinguished if and only if the triangle

$$Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$$

is distinguished.

TR3: For any commutative diagram of the form

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & & & \downarrow{\scriptstyle f[1]} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \end{array}$$

where the rows are distinguished triangles, there is a map $h : Z \to Z'$, which makes the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f[1]} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \end{array}$$

commutative.

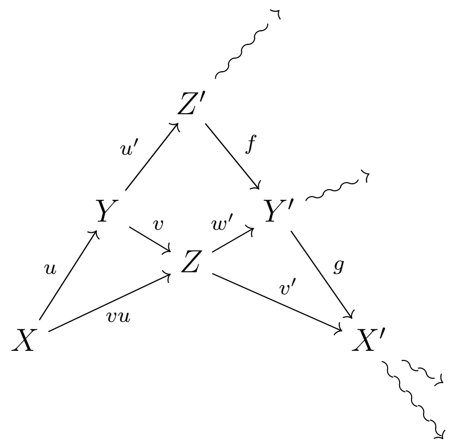
TR4: (Octahedron axiom). Given distinguished triangles

$$X \xrightarrow{u} Y \xrightarrow{u'} Z' \rightsquigarrow,$$
$$Y \xrightarrow{v} Z \xrightarrow{v'} X' \rightsquigarrow,$$
$$X \xrightarrow{vu} Z \xrightarrow{w'} Y' \rightsquigarrow,$$

there exists a distinguished triangle

$$Z' \xrightarrow{f} Y' \xrightarrow{g} X' \rightsquigarrow,$$

such that the following diagram is commutative:



All enclosures in this diagram are commutative, this includes the (not so obvious) square containing the paths from Y' to $Y[1]$ and the square containing the paths from Z' to $X[1]$.

The following is an easy consequence of **TR2**.

Lemma 0.1. *If* $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ *is a distinguished triangle, then*

$$X[-1] \xrightarrow{-u[1]} Y[-1] \xrightarrow{-v[-1]} Z[-1] \xrightarrow{-w[-1]} X[-1]$$

is also a distinguished triangle.

We now observe that the axioms **TR1-TR4** are not wholly independent:

Proposition 0.2. ***TR1**, **TR2** and **TR4** imply **TR3***

Proof. Consider a commutative diagram

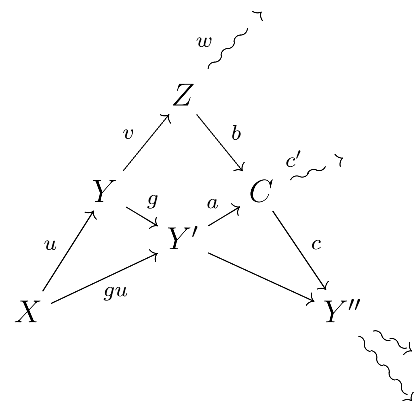
$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\,\vdots & & \downarrow{\scriptstyle f[1]} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \end{array}$$

in which the rows are distinguished triangles, the solid lines indicate morphisms that we are provided with. Then, to prove the proposition we need to construct a morphism h such that the whole diagram commutes. We do this as follows. Complete $X \xrightarrow{gu} Y'$ and $Y \xrightarrow{g} Y'$ to distinguished triangles

$$X \xrightarrow{gu} Y' \xrightarrow{a} C \xrightarrow{c'} X[1] \qquad \text{and} \qquad Y \xrightarrow{g} Y' \longrightarrow Y'' \longrightarrow Y[1].$$

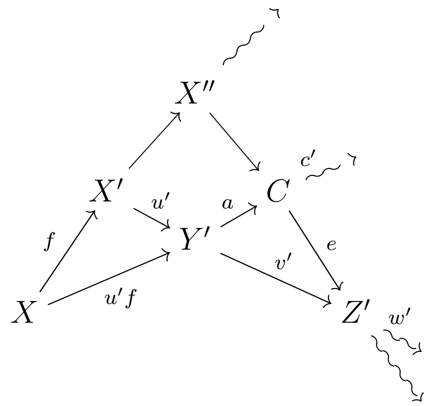
Then applying the octahedron axiom we obtain a commutative diagram

(0.1)



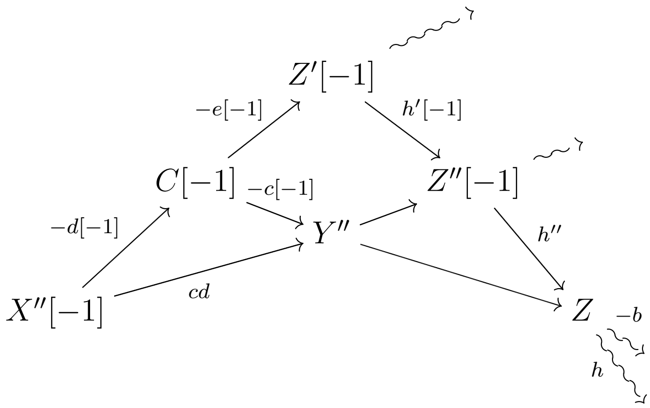
Complete $X \xrightarrow{f} X'$ to a distinguished triangle $X \xrightarrow{f} X' \longrightarrow X'' \rightsquigarrow$, then apply the octahedron axiom to obtain another commutative diagram

(0.2)



Combining Lemma 0.1 with one final application of the octahedron axiom gives the commutative diagram

(0.3)



Now

$$\begin{aligned} hv &= ebv && \text{by the commutativity of (0.3)} \\ &= eag && \text{by the commutativity (0.1)} \\ &= v'g && \text{by the commutativity of (0.2).} \end{aligned}$$

Similarly,

$$\begin{aligned} w'h &= w'cb && \text{by the commutativity of (0.3)} \\ &= f[1]c'b && \text{by the commutativity of (0.2)} \\ &= f[1]w && \text{by the commutativity of (0.1).} \end{aligned}$$

This completes the proof. □

References

[BBD] A. Beilinson, J. Bernstein. P. Deligne, *Faisceaux Pervers*, Asterisque **100** (1982).

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