WEAK $\mathfrak{sl}_2$-CATEGORIFICATIONS

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Put $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $h = ef - fe = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then e, f, h give a basis for the Lie algebra $\mathfrak{sl}_2$.

Let $\mathcal{C}$ be an artinian abelian category with Grothendieck group $K(\mathcal{C})$. Following [CR08], a *weak* $\mathfrak{sl}_2$*-categorification* is the data of an adjoint pair (E, F) of exact endofunctors of $\mathcal{C}$ such that

- the action of $e = [E]$ and $f = [F]$ on $\mathbb{Q} \otimes K(\mathcal{C})$ gives a locally finite $\mathfrak{sl}_2$-representation,
- the classes of the simple objects of $\mathcal{C}$ are weight vectors,
- F is isomorphic to a left adjoint of E.

We put $E_+ = E$, $E_- = F$ and $e_\pm = [E_\pm]$. By the weight space of an object of $\mathcal{C}$, we will mean the weight space of its class (whenever this is meaningful).

Let V be a locally finite representation of $\mathfrak{sl}_2$. Given $\lambda \in \mathbb{Z}$, we denote by V_λ the weight space of V for the weight λ. For $v \in V$ let

$$h_\pm(v) = \max\{n \geq 0 \,|\, e_\pm^n \neq 0\}.$$

Proposition 0.0.1. [CR08, Proposition 5.5] *Fix a weak* $\mathfrak{sl}_2$*-categorification of* $\mathcal{C}$ *and let* $V = \mathbb{Q} \otimes K(\mathcal{C})$*. Let* $\mathcal{C}_\lambda$ *be the full subcategory of objects of whose class is in* V_λ*. Then,* $\mathcal{C} = \bigoplus_\lambda \mathcal{C}_\lambda$*. In particular, the class of an indecomposable object of* $\mathcal{C}$ *is a weight vector.*

Proof. Let L_1 and L_2 be simple objects in different weight spaces. Then, there is $\varepsilon \in \{\pm\}$ and $\{i, j\} = \{1, 2\}$ such that $h_\varepsilon(L_i) > h_\varepsilon(L_j)$. Set $r = h_\varepsilon(L_i)$. Suppose M is an extension of L_1 by L_2. Then, $E_\varepsilon^r M \cong E_\varepsilon^r L_i \neq 0$. So, all the composition factors of $E_{-\varepsilon}^r E_\varepsilon^r M$ are in the same weight space as L_i. Now,

$$\mathrm{Hom}(E_{-\varepsilon}^r E_\varepsilon^r M, M) \cong \mathrm{Hom}(E_\varepsilon^r M, E_\varepsilon^r M) \cong \mathrm{Hom}(M, E_{-\varepsilon}^r E_\varepsilon^r M)$$

and these spaces are not zero. So M has a non-zero simple quotient and a non-zero simple submodule in the same weight space as L_i. Hence, L_i is both a submodule and quotient of M. Consequently $M = L_1 \oplus L_2$.

Thus, $\mathrm{Ext}^1(L_1, L_2) = 0$ whenever L_1 and L_2 are simple objects in different weight spaces. □

References

[CR08] J. Chuang, R. Rouquier, *Derived equivalences for symmetric groups and* $\mathfrak{sl}_2$*-categorification*, Annals of Math. **167** (2008), 245-298.

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