GENERATION AND EQUIVALENCES IN ABELIAN AND TRIANGULATED CATEGORIES

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We will not worry about any kind of set theoretical issues when dealing with categories. We will assume that we remain in a given universe or, as put in [GeMa, p. 38], 'that all the required hygiene regulations are obeyed'.

We will denote the category of sets by **Set**. Unless stated otherwise, all functors will tacitly be covariant.

0.1. **Preliminaries.** Let us recall the notion of a fully faithful functor. Let $\mathcal{A}$ and $\mathcal{B}$ be two categories. A functor $F : \mathcal{A} \to \mathcal{B}$ is *full* if for any two objects $A, B \in \mathcal{A}$ the induced map

$$F : \mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$$

is surjective. The functor F is called *faithful* if this map is injective for all $A, B \in \mathcal{A}$.

Proposition 0.1. *Let $F : \mathcal{A} \to \mathcal{B}$ be a fully faithful functor. Then F is an equivalence if and only if every object $B \in \mathcal{B}$ is isomorphic to an object of the form $F(A)$ for some $A \in \mathcal{A}$.*

Proof. We define an inverse functor F^{-1} as follows: for each $B \in \mathcal{B}$, choose an object $A_B \in \mathcal{A}$ together with an isomorphism $\varphi_B : F(A_B) \xrightarrow{\sim} B$. Then, set $F^{-1}(B) = A_B$ and for $f : B_1 \to B_2$, $F^{-1}(f)$ is given by applying the inverse of the bijection

$$F : \mathrm{Hom}(A_{B_1}, A_{B_2}) \xrightarrow{\sim} \mathrm{Hom}(F(A_{B_1}), F(A_{B_2}))$$

to $\varphi_{B_2}^{-1} \circ f \circ \varphi_{B_1}$. The isomorphisms $FF^{-1} \simeq \mathrm{id}_{\mathcal{B}}$ and $F^{-1}F \simeq \mathrm{id}_{\mathcal{A}}$ are the ones that are naturally induced by the isomorphisms φ_B. □

This immediately yields:

Corollary 0.2. *Any fully faithful functor $F : \mathcal{A} \to \mathcal{B}$ defines an equivalence between $\mathcal{A}$ and the full subcategory of $\mathcal{B}$ of all objects $B \in \mathcal{B}$ isomorphic to $F(A)$ for some $A \in \mathcal{A}$.*

0.2. **Yoneda lemma.** Given a category $\mathcal{A}$, let $\mathcal{A}^{op}$ denote the opposite category. We let $\mathrm{Funct}(\mathcal{A}, \mathbf{Set})$ denote the category of all functors from $\mathcal{A}$ to **Set**. In particular, $\mathrm{Funct}(\mathcal{A}^{op}, \mathbf{Set})$ is the category of contravariant functors from $\mathcal{A}$ to **Set**. Recall that a functor $F \in \mathrm{Funct}(\mathcal{A}^{op})$ is called *representable* if it is isomorphic to $\mathrm{Hom}(-, A)$ for some $A \in \mathcal{A}$.

Lemma 0.3 (Yoneda lemma)**.** *The functor*

$$\Phi : \mathcal{A} \to \mathrm{Funct}(\mathcal{A}^{op}, \mathbf{Set}),$$
$$A \mapsto \mathrm{Hom}(-, A)$$

defines an equivalence of $\mathcal{A}$ with the full subcategory of representable functors $F \in \mathrm{Funct}(\mathcal{A}^{op})$.

Proof. In view of the preceding corollary, it suffices to show that Φ is fully faithful. It is clear that Φ is faithful. Let us show that Φ is full. Let $\varphi : \mathrm{Hom}(-, A) \to \mathrm{Hom}(-, A')$ be a natural transformation. This gives a map $\mathrm{Hom}(A, A) \to \mathrm{Hom}(A, A')$, let ι be the image of id_A under this map. Given $A'' \in \mathcal{A}$ and $f \in \mathrm{Hom}(A'', A)$, we claim that $\varphi(f) = \iota \circ f$ (this will certainly prove that Φ is full). Indeed,

$$\varphi(f) = \varphi(f \circ \mathrm{id}_A) = \varphi(\mathrm{id}_A) \circ f = \iota \circ f.$$

This completes the proof. $\square$

0.3. Adjoint functors. Given categories $\mathcal{A}$ and $\mathcal{B}$, an *adjoint pair* (F^*, F) is the following data: functors $F : \mathcal{A} \to \mathcal{B}$ and $F^* : \mathcal{B} \to \mathcal{A}$, along with two natural transformations

$$\varepsilon : F^*F \to \mathrm{id}_{\mathcal{A}}, \qquad \eta : \mathrm{id}_{\mathcal{B}} \to FF^*,$$

called the *counit* and *unit* respectively, such that the compositions

$$F \xrightarrow{\eta \circ \mathbb{1}_F} FF^*F \xrightarrow{\mathbb{1}_F \circ \varepsilon} F \qquad \text{and} \qquad F^* \xrightarrow{\mathbb{1}_{F^*} \circ \eta} F^*FF^* \xrightarrow{\varepsilon \circ \mathbb{1}_{F^*}} F^*$$

are equal to the identity maps $\mathbb{1}_F : F \to F$ and $\mathbb{1}_{F^*} : F^* \to F^*$, respectively. Given such an adjoint pair (F^*, F), there is an isomorphism functorial in $A \in \mathcal{A}$ and $B \in \mathcal{B}$

$$\alpha_{A,B} : \mathrm{Hom}_{\mathcal{A}}(F^*B, A) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{B}}(B, F(A)), \qquad f \mapsto F(f) \circ \eta(B).$$

The inverse is given by $f' \mapsto \varepsilon_F(F^*F(A)) \circ F^*(f')$. Conversely, the data of such a functorial isomorphism provides the structure of an adjoint pair. Namely, set $\varepsilon(F^*F(A)) = \alpha^{-1}_{A,F(A)}(\mathrm{id}_{F(A)})$ and $\eta(B) = \alpha_{F^*(B),B}(\mathrm{id}_{F^*(B)})$. Note that, by construction, we have commutative diagrams

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{B}}(B, B') & \xrightarrow{F^*} & \mathrm{Hom}_{\mathcal{A}}(F^*(B), F^*(B')) \\ & \searrow^{\eta\circ} & \downarrow{\scriptstyle \alpha_{F^*(B'),B}} \\ & & \mathrm{Hom}_{\mathcal{B}}(B, FF^*(B')) \end{array} \tag{0.1}$$

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}(A, A') & \xrightarrow{F} & \mathrm{Hom}_{\mathcal{B}}(F(A), F(A')) \\ & \searrow^{\circ\varepsilon} & \downarrow{\scriptstyle \alpha^{-1}_{F(A),A'}} \\ & & \mathrm{Hom}_{\mathcal{A}}(F^*F(A), A') \end{array} \tag{0.2}$$

Given an adjoint pair (F^*, F), F^* is said to be *left adjoint* to F and F is said to be *right adjoint* to F^*. The following is a consequence of the Yoneda lemma.

Proposition 0.4. *Suppose a fully faithful functor $F : \mathcal{A} \to \mathcal{B}$ admits a left adjoint F^*. Then the counit*

$$\varepsilon : F^*F \to \mathrm{id}_{\mathcal{A}}$$

is an isomorphism.

Similarly, if a fully faithful functor $F^* : \mathcal{B} \to \mathcal{A}$ admits a right adjoint F, then the unit

$$\eta : \mathrm{id}_{\mathcal{A}} \to FF^*$$

is an isomorphism.

Proof. Assume F is fully faithful. Then the diagram (0.2) shows that

$$\mathrm{Hom}_{\mathcal{A}}(A, A') \xrightarrow{\circ\varepsilon} \mathrm{Hom}_{\mathcal{A}}(F^*F(A), A')$$

is an isomorphism for all $A, A' \in \mathcal{A}$. That is, $\circ\varepsilon : \mathrm{Hom}_{\mathcal{A}}(A, -) \to \mathrm{Hom}_{\mathcal{A}}(F^*F(A), -)$ is an isomorphism in $\mathrm{Funct}(\mathcal{A}, \mathbf{Set})$. It follows from the Yoneda lemma that $\varepsilon : F^*F(A) \to A$ is an isomorphism.

The proof of the second statement is similar. □

Remark 0.5. In short, if (F^*, F) is an adjoint pair, then:

$$F \text{ is fully faithful if and only if } \varepsilon_F : F^*F \xrightarrow{\sim} \mathrm{id}.$$

Similarly,

$$F^* \text{ is fully faithful if and only if } \eta_F : \mathrm{id} \xrightarrow{\sim} FF^*.$$

Proposition 0.6. *Let (F^*, F) be an adjoint pair of functors $F : \mathcal{A} \to \mathcal{B}$, $F^* : \mathcal{B} \to \mathcal{A}$ between abelian categories. Then, F is left exact and F^* is right exact.*

Proof. Suppose $0 \longrightarrow A \xrightarrow{f} A'$ is exact in $\mathcal{A}$. Let us show that $0 \longrightarrow F(A) \xrightarrow{F(f)} F(A')$ is exact in $\mathcal{B}$. By the Yoneda lemma, it suffices to show that

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{B}}(-, F(A)) \xrightarrow{F(f)\circ} \mathrm{Hom}_{\mathcal{B}}(-, F(A'))$$

is exact. Since (F^*, F) is an adjoint pair, for each $B \in \mathcal{B}$ we obtain a commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathrm{Hom}_{\mathcal{B}}(B, F(A)) & \xrightarrow{F(f)\circ} & \mathrm{Hom}_{\mathcal{B}}(B, F(A')) \\ & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ 0 & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(F^*(B), A) & \xrightarrow{f\circ} & \mathrm{Hom}_{\mathcal{A}}(F^*(B), A'). \end{array}$$

The bottom row of this diagram is exact (as $\mathrm{Hom}_{\mathcal{A}}(F^*(B), -)$ is left exact). This forces the top row to also be exact.

The proof of right exactness of F^* is similar. □

Remark 0.7. More generally, given an adjoint pair (F^*, F) (we do not require the functors to be between abelian categories), F^* preserves all colimits and F preserves all limits.

We conclude this subsection with a trivial (but key) observation that is immediate from the defining properties of the unit and the counit.

Lemma 0.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories. Suppose $F : \mathcal{A} \to \mathcal{B}$ is left adjoint to $G : \mathcal{B} \to \mathcal{A}$.*

(i) *If $X \in \mathcal{A}$ is such that $F(X) \neq 0$, then the unit map $\eta : X \to GF(x)$ is not zero.*
(ii) *If $Y \in \mathcal{B}$ is such that $G(Y) \neq 0$, then the counit map $\varepsilon : FG(Y) \to Y$ is not zero.*

0.4. K_0 of an abelian category. Let $\mathcal{A}$ be an abelian category. Define the *Grothendieck group*, denoted $K_0(\mathcal{A})$, as the free abelian group on symbols $[A]$, $A \in \mathcal{A}$, modulo the relation

$$[A] = [A_1] + [A_2]$$

if there is a short exact sequence

$$0 \to A_1 \to A \to A_2 \to 0.$$

A *simple* object in an abelian category $\mathcal{A}$ is an object $L \in \mathcal{A}$ such that any monomorphism $A \to L$ is either 0 or an isomorphism (this automatically implies that any morphism $L \to A$ is either 0 or a monomorphism). Simple objects in $\mathcal{A}$ are also called *objects of length one*. For $n \geq 2$, *objects of length* n are inductively defined to be those objects $A \in \mathcal{A}$ such that there is an exact sequence

$$0 \to A' \to A \to L \to 0$$

with A' of length $n-1$ and L simple. If every object in $\mathcal{A}$ has finite length, then the Jordan-Höleder theorem holds in $\mathcal{A}$ (with the usual proof), i.e., for an object $A \in \mathcal{A}$ of finite length, the length of A is well defined, and the simple objects that occur in a 'composition series' of A are unique upto isomorphism and permutation. The category of finite dimensional representations of an algebra is the standard example of such a category.

Lemma 0.9. *Let $\mathcal{A}$ be an abelian category such that every object in $\mathcal{A}$ has finite length. Let $F, G : \mathcal{A} \to \mathcal{B}$ be exact functors. Suppose $\varepsilon : F \to G$ is a natural transformation which is an isomorphism on simple objects, i.e., $\varepsilon : F(L) \xrightarrow{\sim} G(L)$ for every simple object $L \in \mathcal{A}$. Then $\varepsilon : F \to G$ is an isomorphism.*

Proof. We need to show that $\varepsilon : F(A) \xrightarrow{\sim} G(A)$ for each $A \in \mathcal{A}$. Proceed by induction on the length of A. The base case is given by the statement for simple objects. Assume that the claim is true for objects of length $< n$ and suppose that A is of length n. Then we have an exact sequence $0 \to A' \to A \to L \to 0$, with A' of length $n-1$ and L simple. This gives a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F(A') & \longrightarrow & F(A) & \longrightarrow & F(L) & \longrightarrow & 0 \\ & & \varepsilon \downarrow \wr & & \varepsilon \downarrow & & \varepsilon \downarrow \wr & & \\ 0 & \longrightarrow & G(A') & \longrightarrow & G(A) & \longrightarrow & G(L) & \longrightarrow & 0. \end{array}$$

The outer vertical arrows are isomorphisms by the induction hypothesis. This forces $\varepsilon : F(A) \to G(A)$ to also be an isomorphism. □

Proposition 0.10. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories such that every object in $\mathcal{A}$ and $\mathcal{B}$ has finite length. Let $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{A}$ be functors such that F and G are both left and right adjoint to each other. If F and G induce mutually inverse isomorphisms on Grothendieck groups then, $FG \simeq \mathrm{id}_{\mathcal{B}}$ and $GF \simeq \mathrm{id}_{\mathcal{A}}$.*

Proof. Let $\varepsilon : FG \to \mathrm{id}_{\mathcal{B}}$ be the unit morphism of the adjoint pair (F, G). Lets prove that ε is an isomorphism. By Lemma 0.9 it suffices to show that $\varepsilon : FG(L) \to L$ is an isomorphism for every simple object $L \in \mathcal{B}$. Since $[FG(L)] = [L]$ in $K_0(\mathcal{B})$, we infer that $FG(L) \simeq L$. Consequently, we only need to show that $\varepsilon : FG(L) \to L$ is non-zero. But this is immediate from Lemma 0.8.

The proof of $GF \simeq \mathrm{id}_{\mathcal{A}}$ is similar. □

0.5. Triangulated categories. Let $\mathcal{D}$ be an additive category. The structure of a *triangulated category* on $\mathcal{D}$ is given by the following data:

(i) An additive equivalence $\Sigma : \mathcal{D} \xrightarrow{\sim} \mathcal{D}$. For $X \in \mathcal{D}$ and $n \in \mathbb{Z}$, we will write $X[n]$ instead of $\Sigma^n X$. Further, diagrams of the form

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

will be called *triangles*. A commutative diagram of the form

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & {\scriptstyle f[1]}\downarrow \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z & \xrightarrow{w'} & X'[1] \end{array}$$

will be called a *morphism of triangles*. If f, g, h are isomorphisms then we will say that the two triangles involved are isomorphic.

(ii) A class of *distinguished triangles* satisfying the following axioms:

TR1: For any $X \in \mathcal{D}$ the triangle

$$X \xrightarrow{\text{id}} X \longrightarrow 0 \longrightarrow X[1]$$

is distinguished and any triangle isomorphic to a distinguished one, is itself distinguished. Furthermore, any morphism $X \xrightarrow{u} Y$ can be be completed (not necessarily uniquely) to a distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1].$$

A distinguished triangle

$$X \to Y \to Z \to X[1]$$

will also be written as

$$X \to Y \to Z \rightsquigarrow$$

TR2: (Rotation invariance). A triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

is distinguished if and only if the triangle

$$Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$$

is distinguished.

TR3: For any commutative diagram of the form

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & & & {\scriptstyle f[1]}\downarrow \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \end{array}$$

where the rows are distinguished triangles, there is a map $h : Z \to Z'$, which makes the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle f[1]}\downarrow \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \end{array}$$

commutative.

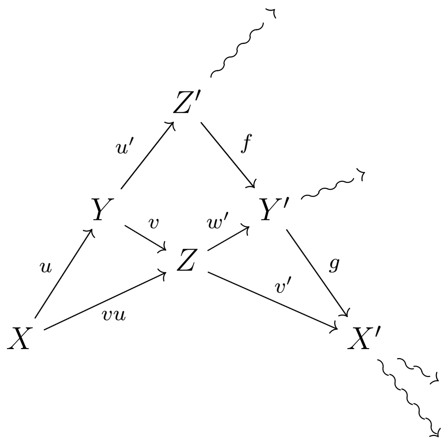
TR4: (Octahedron axiom). Given distinguished triangles

$$X \xrightarrow{u} Y \xrightarrow{u'} Z' \rightsquigarrow,$$
$$Y \xrightarrow{v} Z \xrightarrow{v'} X' \rightsquigarrow,$$
$$X \xrightarrow{vu} Z \xrightarrow{w'} Y' \rightsquigarrow,$$

there exists a distinguished triangle

$$Z' \xrightarrow{f} Y' \xrightarrow{g} X' \rightsquigarrow,$$

such that the following diagram is commutative:



All enclosures in this diagram are commutative, this includes the (not so obvious) square containing the paths from Y' to $Y[1]$ and the square containing the paths from Z' to $X[1]$.

0.6. **Generation for triangulated categories.** Let $\mathcal{D}$ be a triangulated category. Denote by $[\mathcal{D}]$ the collection of isomorphism classes of objects in $\mathcal{D}$, and for $X \in \mathcal{D}$, let $[X]$ denote the corresponding isomorphism class. Let A and B be subcollections of $[\mathcal{D}]$. Following [BBD, §1.3.9], define

$$A*B = \{[Y] \in [\mathcal{D}] \,|\, \text{there is a distinguished triangle } X \to Y \to Z \to X[1] \text{ with } [X] \in A \text{ and } [Z] \in B\}.$$

Lemma 0.11. *The operation $*$ is associative.*

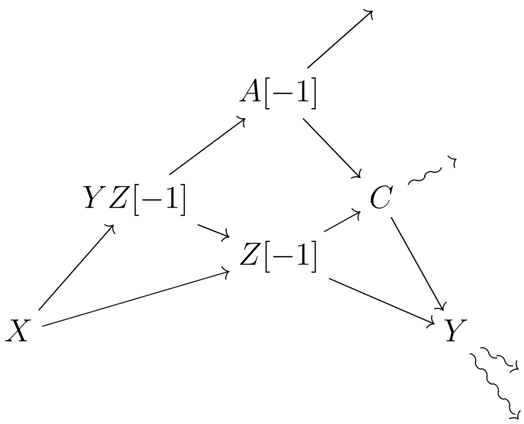
Proof. It suffices to show that for $X, Y, Z \in \mathcal{D}$, one has

$$([X] * [Y]) * [Z] = [X] * ([Y] * [Z]).$$

Suppose A is contained in the left hand side, i.e. we have a diagram

$$\begin{array}{ccccc} & & Y & & \\ & & \downarrow & & \\ X & \longrightarrow A \longrightarrow & YZ & \rightsquigarrow & \\ & & \downarrow & & \\ & & Z & & \\ & & \wr\!\downarrow & & \end{array}$$

The octahedral axiom then gives us a commutative diagram



We infer that A is contained in the right hand side. The reverse inclusion is proved similarly. □

Let $\mathcal{I}$ be a subcategory of $\mathcal{D}$. Denote by $\langle\mathcal{I}\rangle$ the smallest full subcategory of $\mathcal{D}$ containing $\mathcal{I}$ and closed under finite direct sums, direct summands and shifts. Put $\langle\mathcal{I}\rangle_0 = 0$ and inductively define $\langle\mathcal{I}\rangle_i = \langle\mathcal{I}\rangle_{i-1} * \langle\mathcal{I}\rangle$, $i \geq 1$. We put $\langle\mathcal{I}\rangle_\infty = \bigcup_{i\geq 0}\langle\mathcal{I}\rangle_i$. We say that $\mathcal{I}$ *generates* $\mathcal{D}$ if every object in $\mathcal{D}$ is isomorphic to some object in $\langle\mathcal{I}\rangle_\infty$. Further, we say that $X \in \mathcal{D}$ is of length n (relative to $\mathcal{I}$) if n is minimal with the property that X is isomorphic to some object in $\langle\mathcal{I}\rangle_n$.

The following result is the triangulated analogue of Lemma 0.9.

Proposition 0.12. *Let $\mathcal{I}$ be a generating subcategory of $\mathcal{D}$. Let $F, G : \mathcal{D} \to \mathcal{D}'$ be triangulated functors. Suppose $\varepsilon : F \to G$ is a natural transformation which is an isomorphism on $\mathcal{I}$, i.e., $\varepsilon : F(L) \xrightarrow{\sim} G(L)$ for every $L \in \mathcal{I}$. Then ε is an isomorphism.*

Proof. We need to show that $\varepsilon : F(X) \xrightarrow{\sim} G(X)$ for each $X \in \mathcal{D}$. Proceed by induction on the length (relative to $\mathcal{I}$) of X. The base case is given by the statement for objects in $\mathcal{I}$. Assume that the result holds for objects of length $< n$ and suppose X is of length n. Then we have a distinguished triangle $X' \to X \to L \rightsquigarrow$, with X' of length $n-1$ and $L \in \mathcal{I}$. This gives a morphism of triangles

$$\begin{array}{ccccccc} F(X') & \longrightarrow & F(X) & \longrightarrow & F(L) & \rightsquigarrow \\ \varepsilon \downarrow \wr & & \varepsilon \downarrow & & \varepsilon \downarrow \wr & \\ G(X') & \longrightarrow & G(X) & \longrightarrow & G(L) & \rightsquigarrow \end{array}$$

By the induction hypothesis, the outer vertical arrows are isomorphisms. This forces $\varepsilon : F(X) \to G(X)$ to also be an isomorphism. $\square$

References

[BBD] A. A. Beilinson, J. Bernstein, P. Deligne, *Faisceaux pervers*, Analyse et topologie sur les espaces singulares, Asrérisque **100** (1982), 1-171.

[GeMa] S. I. Gelfand, Y. I. Manin, *Methods of Homological Algebra*, Springer Monographs in Mathematics, New York (second edition 2003).

[KaSc] M. Kashiwara, P. Schapira, *Categories and Sheaves*, Grundlehren der mathematischen Wissenschaften **332**, Springer-Verlag, Berlin (2006).

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