The passage from $\mathbb{F}$ to $\mathbb{C}$ à la [BBD] (W. Soergel)

(Freiburg, Jul 2012) 1

$f : X \longrightarrow Y$ proper algebraic map of varieties $/_{\mathbb{C}}$

$\mathcal{F}$ - simple perverse sheaf on X of geometric origin

$\Rightarrow Rf_* \mathcal{F}$ perversely semi-simple complex on Y

w/ coefficients in a field of char 0.

1) $\underbrace{\text{without loss of generality}}$ we can assume $X, Y, \mathcal{F}$ — or just assume we are in this situation

is defined over $A \subset \mathbb{C}$

$\uparrow$ finite ~~type~~ $/\mathbb{Z}$

$X/_{\mathbb{C}}$, maybe $X \subset \mathbb{P}^n(\mathbb{C})$, $\sum a_\alpha x^\alpha = 0$ $\quad a_\alpha \in \mathbb{C}$

$\uparrow$ defined over

$S = \operatorname{spec}(\mathbb{Z}[a_\alpha])$

$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d] \\
s \arrow[r, hook] & S
\end{tikzcd}$$

$\exists$ stratification of base, s.t. topological type of fibres is constant on strata.

$X/_A$ - fix stratification $\mathcal{T}$, fix local systems on strata (finitely many) $\mathcal{L}$'s.

$$\begin{array}{ccccc} \mathbb{C} & \supset & A & \supset & \mathfrak{m} \\ & \cup & & \cap & \\ & & \mathcal{V} & & \end{array}$$

DVR, strictly henselian

means to be contractible in the etale topology

$$\begin{array}{ccc} A & \longrightarrow & A/\mathfrak{m} \\ \cap & & \\ \mathcal{V}/\mathfrak{m} & & \end{array}$$

algebraic closure

$$\begin{tikzcd} D^b_{T,\mathcal{L}}(X_\mathcal{V}, \mathbb{Z}_\ell) \arrow[r, "j^*", "\sim"'] \arrow[d, "i^*"'] & D^b_{T,\mathcal{L}}(X, \mathbb{Z}_\ell) \\ D^b_{T,\mathcal{L}}(X_F, \mathbb{Z}_\ell) & \end{tikzcd}$$

$$X_\mathbb{C} \xhookrightarrow{j} X_\mathcal{V} \xhookleftarrow{i} X_\mathbb{F}$$

$\pi : Y \longrightarrow X$ semi-small proper

$$\operatorname{codim} \{ x \in X \mid \dim_\mathbb{C} \pi^{-1}(x) \geq n \} \geq 2n$$

assume smooth

$\Rightarrow$ $R\pi_* \underline{Y}[d_Y]$ is perverse

as the stalks of are cohomology of fibres

now use the semismall assumption.

$\rightsquigarrow R\pi_* \underline{Y}[d_Y] = \bigoplus IC_S$

the skyscraper pieces will be ones where fibres are of max dim.