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The Hodge theorem as a derived self intersection

j/ work w/ D. Arinkin, M. Hablicsek.

k - field ; char $k = p > 0$

$$\Omega^\bullet : 0 \longrightarrow k[x] \xrightarrow{d} k[x]\,dx \longrightarrow 0$$

$$d : f \longmapsto df$$

$$H^0(\Omega^\bullet) = k[x^p]$$

$$H^1(\Omega^\bullet) = k[x^p]\,x^{p-1}\,dx$$

$$\begin{tikzcd}
0 \arrow[r] & k[x^p] \arrow[r, "0"] \arrow[d, hook] & k[x^p]\,d(x^p) \arrow[r] \arrow[d, hook, "f\,d(x^p) \mapsto f x^{p-1} dx"] & 0 \\
0 \arrow[r] & k[x] \arrow[r, "d"] & k[x]\,dx \arrow[r] & 0
\end{tikzcd}$$

this map is a qis.

$\Omega^\bullet$ is complex of $k[x^p]$-modules.

So $\Omega^\bullet$ is a split complex of $k[x^p]$-modules

($A^\bullet$ is split if $A^\bullet \simeq \bigoplus H^i(A^\bullet)[-i]$)

(should remind you of Hodge theory on Kähler manifolds, see Griffiths - Harris)

$\Omega^\bullet$ makes sense for any scheme (Grothendieck)

$$0 \to \Omega^0_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \Omega^2_X \to \cdots$$

algebraic de Rham complex

regard it as a complex of sheaves over

$X' =$ Frobenius twist of X $\quad (\underset{\substack{\uparrow \\ p^{th}\text{-power}}}{\mathcal{O}_X'} \subseteq \mathcal{O}_X)$

$\overset{\displaystyle F_* \Omega^\bullet_X .}{\underset{\text{Frobenius map}}{\uparrow}}$

<u>Question</u> If X is smooth$/_k$ does $F_*\Omega^\bullet_X$ split?

<u>Cartier</u> $H^i(F_*\Omega^\bullet_X) = \Omega^i_{X'}$

<u>Reformulate question</u>: when is

$$F_*\Omega^\bullet_X \simeq \bigoplus_i \Omega^i_{X'}[-i] \,?$$

<u>Hypercohomology spectral sequence</u>

$$E_1^{p,q} = H^p(X; \Omega^p_X) \Rightarrow R\Gamma^{p+q}(\Omega^\bullet_X)$$
$$= H^{p+q}_{dR}(X).$$

<u>Deligne - Illusie</u> If the answer to the question is yes, then the above spectral sequence degenerates at E_1 and

$$H^n_{dR}(X) = \bigoplus_{p+q=n} H^p(X; \Omega^q)$$

Thm (Deligne - Illusie 88-90)

If X is smooth $/_k$, char $k = p > \dim X$, and X lifts to $W_2(k)$, then $F_* \Omega^\bullet_X$ splits.

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Example of another complex which naturally splits

$X = Z(f,g) \subseteq Y$ $\qquad$ X is a global complete intersection

$i : X \hookrightarrow Y$

$Li^* i_* \mathcal{O}_X$

say $\mathcal{O}_Y = R$; $\mathcal{O}_X = R/_{\langle f, g \rangle}$

→ as complete intersection, Koszul complex gives a resolution of $\mathcal{O}_X$

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_Y \arrow[r, "\begin{pmatrix} g \\ -f \end{pmatrix}"'] & \begin{matrix} \mathcal{O}_Y \\ \oplus \\ \mathcal{O}_Y \end{matrix} \arrow[r, "(f, g)"'] & \mathcal{O}_Y \arrow[r] & 0 \end{tikzcd}$$

$- \otimes \mathcal{O}_X$ to get

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_X \arrow[r, "0"] & \begin{matrix} \mathcal{O}_X \\ \oplus \\ \mathcal{O}_X \end{matrix} \arrow[r, "0"] & \mathcal{O}_X \arrow[r] & 0 \end{tikzcd}$$

~~to computes $Li^* i_*$~~

computes $Li^* i_* \mathcal{O}_X$

<u>Question</u> If $i : X \hookrightarrow Y$ is a closed embedding of smooth schemes, when is $i^* i_* \mathcal{O}_X$ split?

↑ derived from now on.

$$H^j(i^* i_* \mathcal{O}_X) \simeq \wedge^j \underbrace{N^\vee_{X/Y}}_{\text{conormal bundle}}$$

So the question is when is

$$i^* i_* \mathcal{O}_X \simeq \bigoplus_j \wedge^j N^\vee_{X/Y}[j] \ ?$$

<u>Thm (Arinkin, –)</u> ~~$i^* i_* \mathcal{O}_X$ splits if and only if char $k = 0$ or $>$ $\mathrm{codim}_Y X$~~

Assume char $k = 0$ or $>$ $\mathrm{codim}_Y X$. Then

$i^* i_* \mathcal{O}_X$ splits $\Longleftrightarrow$ N extends to 1st infinitesimal nbhd. of X inside Y.

↑ normal bundle of X in Y

<u>Eg</u> If $X \xrightarrow{\Delta} X \times X$, then the condition is satified and

$$\Delta^* \Delta_* \simeq \bigoplus_i \Omega^i_X[i]$$ is the HKR isomorphism

(contd.) if E is a vector bundle on X, then $i^* i_* E$ splits $\Longleftrightarrow$ both E and N extend to 1st infinitesimal nbhd. of X in Y

- In Deligne-Illusie lifting to $W_2(k)$ "is" lifting to 1st infinitesimal nbhd.
- Lots of similarities w/ Deligne-Illusie theorem

* **Question** (Mustata) Given a smooth X, can we find $X' \xrightarrow{i} Y$ (some Y) s.t. splitting of $i^*i_*\mathcal{O}_{X'} \Rightarrow$ splitting of $F_*\Omega^\bullet_X$?

Thm If X is smooth$/k$, char $k > 0$, consider the closed embedding of twisted spaces

$$\begin{tikzcd}
(X', \mathcal{O}_{X'}) \arrow[r, "\sim"] \arrow[rr, bend right, "i"'] & (X', \mathcal{D}|_{X'}) \arrow[r, hook] & (\mathrm{Tot}\,\Omega^1_{X'}, \mathcal{D})
\end{tikzcd}$$

Then $i^*i_*\mathcal{O}_{X'} \simeq (F_*\Omega^\bullet_X)^\vee$

Moreover, $\mathcal{O}_{X'}$ and N extend to the 1st infinitesimal nbhd $\Leftrightarrow$ X lifts to $W_2(k)$.

$\mathcal{D}$ = ring of differential operators on X.

$Z(\mathcal{D}) = \mathcal{O}_{\mathrm{Tot}\,\Omega^1_{X'}}$ in char p

$X = \mathrm{Spec}\; k[x]$

$\mathcal{D}$ = Weyl alg. $k[x, \partial]$

$Z(\mathcal{D}) = k[x^p, \partial^p]$ in char p

$= \mathcal{O}_{\Omega^1_{X'}}$

$\mathcal{D}$ is Azumaya $/ Z(\mathcal{D})$

What about twisted diff. ops?