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Tensor triangulated categories

General picture

Tensor triangulated geometry:

Examples come from:

Algebraic geometry

Topology

modular representation theory

motivic theory

Non-commutative geometry

K - triangulated category w/ bifunctor $\otimes$ and unit $\mathbb{1}$. & strict monoidal.

$a \otimes -$ is exact.

Assume K is essentially small.

* Algebraic geometry example. X quasi-compact + quasi separated. $K = D^{per}(X)$.
* Topology example: stable homotopy category
* Modular representation theory: stable module category of a finite group.

* non-commutative geometry : C^*-algebras, rings of continuous functions $C(X, \mathbb{C})$

naive notion of homotopy due to $[0,1]$

K = thick subcategory generated by $\mathbb{1}$

Geometry?

$(K, \otimes, \mathbb{1})$: ~~specK~~ $\mathrm{spec}(K) \supseteq \mathrm{supp}(a)$, $a \in K$

closed

require 'obvious' properties

$\mathrm{supp}(0) = \phi$; $\mathrm{supp}(\mathbb{1}) = \mathrm{spec}(K)$

$\mathrm{supp}(a \oplus b) = \mathrm{supp}(a) \cup \mathrm{supp}(b)$

if $a \to b \to c \rightsquigarrow$ a d.t. then

$\mathrm{supp}(c) \subset \mathrm{supp}(a) \cup \mathrm{supp}(b)$

$\mathrm{supp}(a \otimes b) = \mathrm{supp}(a) \cap \mathrm{supp}(b)$

some version of Nakayama's lemma

can actually make a locally ringed space

$(\mathrm{spec}(K), \mathcal{O}_K)$

if X quasi compact, quasi ~~sp~~ separated scheme then

$\mathrm{spec}(D^{\mathrm{per}}(X)) \cong X$

In modular rep. theory

$$\mathrm{spec}(kG\text{-stab}) \cong \mathcal{V}_G(k) = \mathrm{Proj}(H^\cdot(G,k))$$

<u>Digression</u>

$$\mathrm{spec}(K) = \{ P \subsetneq K \mid \text{thick} \otimes \text{prime ideal} \}$$

→ original motivation for definition was <u>NOT</u> classical commutative ring theory, but rather required ~~universal~~ 'sensible universal' properties of spec(K). Point is that really the notion of <u>support</u> is the important one.

Construction:

$$U \subseteq \mathrm{spec}(K) \qquad Z = \mathrm{spec}(K) \setminus U$$

$$K_Z \hookrightarrow K \longrightarrow K/K_Z \hookrightarrow \underbrace{(K/K_Z)^{\natural}}_{\text{idempotent completion}} := K(U)$$

Thomason: if $K = D^{perf}(X)$, Thomason $K(U) \simeq D^{per}(X)$

Endotrivial modules

setting of modular rep. theory

$char(k) \mid |G|$

endotrivial module M s.t

$End_k(M) = k \oplus$ projectives

i.e $End_k(M)$ is invertible in the stable module category.

study $Pic(K)$, i.e isomorphism classes of invertible objects in K.

Let $T(G) = Pic(kG\text{-stab})$

Thm $T(G) \otimes_{\mathbb{Z}} \mathbb{Q} \cong Pic(V_G(k)) \otimes \mathbb{Q}$

Carlson-Nakano: Is this true for G finite group schemes?

$$Pic_{e.t}(K) \subseteq Pic(K)$$

$$\| \quad \{a \in Pic(K) \mid \exists \text{ open cover of } Spec(K) = \bigcup_i U_i \text{ s.t. } res_{U_i}(a) \text{ is trivial for all } i\}$$

$$res_{U_i}(a) = \mathbb{1}$$

$a = \otimes$ [illegible]

$G = Q_8$ $\quad$ char $k = 2$

$\mathcal{V}_{Q_8} = \{pt\}$

$T(Q_8) = \underbrace{\mathbb{Z}/4\mathbb{Z}}_{\text{suspension}} \oplus \mathbb{Z}/2\mathbb{Z}$

<u>Def</u>/<u>Thm</u> $\quad$ K local if spec(K) local top. space

$\iff$ spec(K) unique closed pt.

$\iff$ $a \otimes b = 0 \Rightarrow a = 0$ or $b = 0$.

<u>question</u>

u invertible $\overset{?}{\Longrightarrow}$ $\exists m > 0$ s.t. $u^{\otimes m}$ is trivial ($= \Sigma^n \mathbb{1}$).