S. Kovacs   Generic vanishing fails for singular varieties in characteristic $p>0$ ①

$$\mathbb{C} \xrightarrow{\exp} \mathbb{C}^*$$
$$z \longmapsto e^{2\pi i z}$$

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \longrightarrow 0$$

$\rightsquigarrow$ $$0 \longrightarrow \mathbb{Z}_X \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 0$$

$\rightsquigarrow$ long exact sequence in cohomology

$\rightsquigarrow$ $$0 \longrightarrow \underset{\mathbb{Z}^{2g}}{\overset{}{H^1(X;\mathbb{Z}_X)}} \longrightarrow \underset{\mathbb{C}^g}{H^1(X;\mathcal{O}_X)} \longrightarrow \underset{\mathrm{Pic}(X)}{H^1(X;\mathcal{O}_X^*)}$$
$$\longrightarrow H^2(X;\mathbb{Z}_X) \longrightarrow \cdots$$

$$H^1(X;\mathcal{O}_X^*) \longrightarrow H^2(X;\mathbb{Z}_X)$$
$$\mathcal{L} \longmapsto c_1(\mathcal{L})$$

$$g = h^{0,1} = h^{1,0}$$
$$b_1 = h^{0,1} + h^{1,0} = 2g$$

$$\mathrm{Pic}^0(X) = \ker c_1 \simeq \mathbb{C}^g/\mathbb{Z}^{2g}$$ } complex torus

X - smooth proj. variety $\Longrightarrow$ abelian variety

$X = A$ (abelian variety itself)

then $\mathrm{Pic}^0(A) = \hat{A}$ (dual abelian variety)

$\hat{\hat{A}} = A$ and $\dim(\hat{A}) = \dim(A)$

X - smooth projective variety.

$\alpha : X \longrightarrow A$

$H^0(X; \Omega^1_X)$

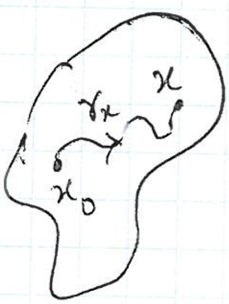
$$H_1(X; \mathbb{Z}_X) \longrightarrow H^0(X_1; \Omega^1_X)^* \longrightarrow \mathbb{C}^g/\mathbb{Z}^{2g}$$

$$\gamma \longmapsto \int_\gamma -$$

$$\uparrow$$

$$\mathrm{Alb}(X)$$

Fix $x_0 \in X$



$$X \longrightarrow \mathrm{Alb}(X)$$

$$x \longmapsto \int_{\gamma_x} - \in H^0(X, \Omega^1_X)^*$$

$\uparrow$

take equivalence class $\left[\int_{\gamma_x} -\right]$

Stokes' theorem!

Universal property

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "alb_X"'] & A \text{ (abelian variety)} \\ & \mathrm{Alb}(X) \arrow[u, dashed] \end{tikzcd}$$

Albanese dimension

$$\dim \cancel{\mathrm{Alb}}\ \mathrm{alb}_X(X) \leq \dim X$$

If X is abelian, then $X = \mathrm{Alb}(X)$

$$X \longrightarrow \mathrm{Alb}(X)$$

$$\widehat{\mathrm{Alb}}(X) = \mathrm{Pic}^0(X)$$

$$\mathcal{P} \in A \times \hat{A}$$

normalized Poincare line bundle.

$$\forall \hat{a} \in \hat{A},\ [\mathcal{P}|_{A \times [\hat{a}]}] = \hat{a}$$

look at [Birkenhake - Lange]

take obvious correspondence $\subset A \times \hat{A}$

$$(x, \mathcal{L}_x)$$

$$H^i(A, \mathcal{L}) = 0 \quad \forall i \qquad \mathcal{L} \neq \mathcal{O}_A,\ \mathcal{L} \in \mathrm{Pic}^0 A$$

Thm (Mumford) $g = \dim A$

$$\begin{tikzcd} & A \times \hat{A} \arrow[dl, "p_A"'] \arrow[dr, "p_{\hat{A}}"] & \\ A & & \hat{A} \end{tikzcd}$$

$$R^i p_{\hat{A}*} \mathcal{P} = \begin{cases} 0 & \text{if } i < g \\ k(0) & \text{if } i = g \end{cases}$$

$0 \in \mathrm{Pic}^0(A)$ (identity)

Green – Lazarsfeld

X – has maximal Albanese dimension
$\iff$ ~~gen~~ admits generically finite map to an abelian variety
$\alpha : X \longrightarrow A$

$$\begin{tikzcd}
 & X \times \hat{A} \arrow[dl, "\pi_X"'] \arrow[d, "\alpha \times id"] \arrow[rr, "\pi_{\hat{A}}"] & & \hat{A} \arrow[dl, "id"] \\
X \arrow[r, "\alpha"'] & A & A \times \hat{A} \arrow[l, "p_A"'] \arrow[r, "p_{\hat{A}}"] & \hat{A}
\end{tikzcd}$$

$$\mathcal{L} = (\alpha \times id)^* \mathcal{P}$$

Conjecture (Green – Lazarsfeld) '91

$$R^i \pi_{\hat{A}*} \mathcal{L} = 0 \qquad \text{for } i < \dim X$$

Proved

Hacon '04 for X – projective

Pareschi – Popa '09 for X – Kähler

Generalizations a) for X – singular

b) for char $= p > 0$

trivial

same is true ~~for~~ if X has rational singularities

$\sigma : \tilde{X} \to X$ res. of singularities
s.t. $R\sigma_* \mathcal{O}_{\tilde{X}} \simeq \mathcal{O}_X \iff X$ is normal $+ R^i \sigma_* \mathcal{O}_{\tilde{X}} = 0$

Pareschi '03 (preprint)

worry about truth of this result.

X is Gorenstein and char. is arbitrary

Assume that α is separable

$\Rightarrow R^i\pi_{\hat{A}*}\mathcal{L} = 0$ for all $i < \dim X$

X has Gorenstein singularities if $\omega_X \simeq \omega_X^\cdot[n]$

$n = \dim X$ and ω_X is a line bundle

<u>example</u> locally complete intersection.

rational $\Rightarrow$ Cohen-Macaulay

$\Uparrow$

Gorenstein

<u>Thm</u> (Hacon, K.)

a) char $= 0$ $\Rightarrow \exists X$ a projective variety w/ isolated Gorenstein log-canonical singularities

b) char ~~$\ne 0$~~ $= p > 0$ $\Rightarrow \exists X$ a smooth projective variety s.t. $alb_X : X \longrightarrow Alb(X)$ is generically finite separable and $R^i\pi_{\hat{A}*}\mathcal{L} \neq 0$ for some $0 \le i < \dim X$.

Key Proposition

If the desired vanishing $R^i\pi_{\hat{A}*}\mathcal{Z} = 0$ $\forall i < \dim$ holds $\Rightarrow R^i\alpha_*\omega_X = 0$ $\forall i > 0$

Grauert - Riemenschneider vanishing thm

$f : Z \longrightarrow Y$ proper birational morphism

Z is smooth and char $= 0$

$\Rightarrow R^i f_* \omega_Z = 0$ $\quad i > 0$

Construction :

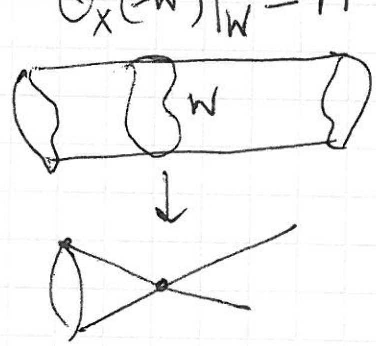
Step 1 : construct a local map w/ the desired non vanishing property

Step 2 : prove that the target maps generically finitely to an abelian variety.

Step 1 is easier for b) (char p) : Take W - smooth projective variety s.t Kodaira vanishing fails for W.

$Y = \overline{C(W)}$

$\mathcal{O}_X(-W)|_W \simeq H$



$X = Bl(Y)$

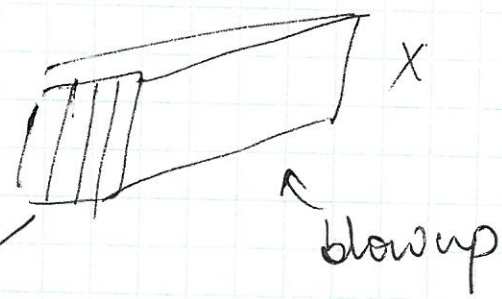
$\exists$ ~~line~~ very ample line bundle H on W s.t. $H^i(W; \omega_W \otimes H) \neq 0$ for some $i > 0$

Aside : generically finite map from a curve is finite.

Step 1 for A) (char 0)

$E_1 \times E_2$

elliptic curves



Y = cone over $E_1 \times E_2$