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Geometric Realizations of quantum groups

categorification

↙ ↘

algebraic   geometric

given graph, break it up into 2 types:

$E_m = \{$ nilpotent $m \times m$ matrices $\}$

$Gl_m \curvearrowright E_m$; orbits $\longleftrightarrow$ partitions of m

$\bigoplus_m \mathbb{C}(E_m) \xrightarrow{\sim} \mathbb{C}[x_1, x_2, \dots]$

$\xrightarrow{\sim} \bigoplus_m K_0(S_m\text{-mod})$

'no loops'

Γ

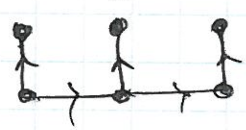
$E_v \rightsquigarrow U^- = U_\Gamma$

IC $\rightsquigarrow$ canonical basis

$\mathrm{Ext}^\bullet(\oplus \mathrm{IC}, \oplus \mathrm{IC})$

$\xrightarrow{\sim}$ KLR

$\tilde{\Gamma}$ - framed graph of Γ



$E_{v,d} \xrightarrow{+\text{ stability}} V_{\lambda_1}, \; V_{\lambda_1} \otimes V_{\lambda_2}$

w/o stability $\longrightarrow M_0 \otimes V_\lambda$

consider

$$E_{m,d} = E_m \times \mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^d)$$

$$\mathcal{B}_m = \{\mathbb{C}^m = V_1 \supset V_2 \supset \cdots \supset V_m = 0\}$$ complete flag variety

fix $\underline{d} = (d_1, \ldots, d_N)$ a composition of d

and a flag $\underline{D}$ of type $\underline{d}$

$$\underline{D} = \{\mathbb{C}^d = D_1 \supset D_2 \cdots \supset D_N = 0\}$$

w/ $\dim D_i / D_{i+1} = d_i$

$\underline{m} = (m_1, \ldots, m_N)$ a composition of m

$$\tilde{\mathcal{B}}_m = \left\{ (\underline{V}, x, q) \;\middle|\; \begin{array}{l} x(V_i) \subseteq V_{i+1} \\ q(V_{m_1+1}) \subseteq D_1 \\ q(V_{m_1+m_2+1}) \subseteq D_2 \end{array} \right\} \subseteq \mathcal{B}_m \times E_{m,d}$$

$$\begin{tikzcd} & \tilde{\mathcal{B}}_m \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{B}_m & & E_{m,d} \end{tikzcd}$$

vector bundle $\longrightarrow$ π_1

π_2 — proper actually projective and semi small

$$Z_{\underline{m}} = \tilde{\mathcal{B}}_{\underline{m}} \times_{E_{m,d}} \tilde{\mathcal{B}}_m \longrightarrow \tilde{\mathcal{B}} \times_{E_m} \tilde{\mathcal{B}}_m \longrightarrow \underbrace{\mathcal{B}_m \times \mathcal{B}_m}_{\substack{U_w \\ \text{(orbits)}}}$$

Y_w

steinberg variety

1) $d = (0)$ original springer resolution

2) $d = (1,0)$

$E_{m,d} = E_m \times \mathbb{C}^m$ enhanced nilpotent cone

set:

$$L_{\underline{m}} = \pi_{2!}\left(\mathbb{C}_{\tilde{\mathcal{B}}_{\underline{m}}}\right)\left[d_{\tilde{\mathcal{B}}_{\underline{m}}}\right]$$

$=$ semisimple perverse sheaf

$$= \bigoplus_{\lambda,\alpha} IC(E(\lambda),\alpha) \otimes W_{\lambda,\alpha}$$

multiplicity.

α is trivial

$E(\lambda)$ can be described.

$$\lambda = (\lambda_1, \ldots, \lambda_N) \qquad |\lambda_1| + \cdots + |\lambda_m| = m$$

$$\mathcal{B}_m \longrightarrow \mathcal{P}_\lambda$$

$$E(\lambda) = \operatorname{im}(\pi_\lambda)$$

$$\mathcal{P}_\lambda \xleftarrow{\alpha} \tilde{\mathcal{P}}_\lambda \xrightarrow{\pi_\lambda} E_{m,d}$$

$$W_\lambda = V_{\lambda_1} \otimes V_{\lambda_2} \otimes \cdots \otimes V_{\lambda_N}$$

$$S_{m_1} \times \cdots \times S_{m_N}$$

V_{λ_i} s are simple S_{m_i} modules

$$\mathrm{End}(L_{\underline{m}}, L_{\underline{m}}) \simeq \underbrace{H_{top}}_{\text{Borel-Moore}}(Z_{\underline{m}}) \xrightarrow{\sim} \mathbb{C}[S_{m_1} \times \cdots \times S_{m_N}]$$

Tensor product schur algebra

singular support ~~SS~~ SS contained in $T^*_{E_{m,d}}$