Rook placements and Jordan forms M. Yip

30 April 2013 (CU-Boulder)

$\mathfrak{g}_n(\mathbb{F}_q)$ - $n \times n$ upper triangular nilpotent matrices

$\lambda \vdash n$ a partition

$$\mathfrak{g}_{n,\lambda}(\mathbb{F}_q) = \{ x \in \mathfrak{g}_n(\mathbb{F}_q) \mid x \sim J_\lambda \}$$

Question:

Calculate $F_\lambda(q) = |\mathfrak{g}_{n,\lambda}(\mathbb{F}_q)|$

Example $n = 3$

$$F_{(3)}(q) = (q-1)^2 q \quad \text{(rank 2)}$$

$$F_{(2,1)}(q) = q^3 - (q-1)^2 q - 1 = (q-1)(2q+1)$$

$$F_{(1,1,1)}(q) = 1 \quad \text{(zero matrix).}$$

Recurrence formula

$$F_\lambda(q) = \sum_{\mu \lessdot \lambda} c_{\lambda,\mu}(q) \, F_\mu(q)$$

$$c_{\lambda,\mu}(q) = \begin{cases} q^{|\mu| - \mu_j'} & j = 1 \\ q^{|\mu| - \mu_{j-1}'} \left(q^{\mu_{j-1}' - \mu_j'} - 1 \right) & j \geq 2 \end{cases}$$

$$F_\lambda(q) = \sum_{\substack{\text{paths} \\ \text{in Young's} \\ \text{lattice}}} \prod \text{edge weights}$$

- $F_\lambda(q) \in \mathbb{Z}[q]$
- $\deg F_\lambda(q) = \binom{n}{2} - n(\lambda)$ $\qquad n(\lambda) = \sum_{i \geq 1} (i-1)\lambda_i$
- wt. of each path to λ has same degree

$[q^{\binom{n}{2} - n(\lambda)}]\, F_\lambda(q) = f^\lambda$ (# of standard Young tableaux of shape λ).

<u>observation</u>: $F_\lambda(q) = (q-1)^k$ (poly. in q w/ non-negative integer coefficients)

<u>connection</u> w/ <u>rook placements</u>

Garsia-Remmel

$$\sum_{k=1}^{n} \frac{[k]_q!\, S_{n,k}(q)\, x^k}{(1-x)(1-qx)\cdots(1-q^k x)}$$

$$= \sum_{\sigma \in S_n} \frac{x^{des(\sigma)+1}\, q^{maj(\sigma)}}{(1-x)(1-qx)\cdots(1-q^n x)}$$

$[k] = 1 + q + \cdots + q^{k-1}$

$\sigma = \sigma_1 \cdots \sigma_m$

$des(\sigma) = \{ i \mid \sigma_i > \sigma_{i+1} \}$

$maj(\sigma) = \sum_{i \in des(\sigma)} i$

B - Ferrers board

$C(B,k)$ - set of placements of k non-attacking rooks on B

$\mathrm{inv}(c) = \mathrm{area}(B) - \#\text{rooks} - \#\text{NE squares}$

The q-rook polynomial is

$$R_{B,k}(q) = \sum_{c \in C(B,k)} q^{\mathrm{inv}(c)}$$

Solomon:

$$\left\{\begin{matrix}\text{placements of } k \text{ rooks} \\ \text{on } m \times n \text{ board}\end{matrix}\right\} \xleftrightarrow{1-1} \left\{\begin{matrix}m \times n\text{-matrices}/\mathbb{F}_q \\ \text{w/ rank } k\end{matrix}\right\}$$

Haglund:

$$P_{\lambda,k} = (q-1)^k q^{\mathrm{area}(B)-k} R_{B,k}(q^{-1})$$

staircase

$$= \sum_{c \in C(\partial_n, k)} (q-1)^{\#x} q^{\#\cdot}$$

$$= \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) = n-k}} F_\lambda(q).$$

Theorem [Y]

$$C(\partial_n, k) \xleftrightarrow{1-1} \left\{\begin{matrix}\text{paths in Young lattice } \phi \to \lambda \\ \text{satisfying } \ell(\lambda) = n-k\end{matrix}\right\}$$

$$c \longleftrightarrow p$$

wt. of i^{th} column in c = wt. of i^{th} edge in p.

The partition type of a rook placement c is the endpoint of p.

<u>Cor</u> $F_\lambda(q) = \sum_{\substack{\text{placements} \\ \text{of } \cancel{\text{type}}\ \lambda}} (q-1)^{\#x} q^{\#\bullet}$

Some concluding comments on Hall-Littlewood polynomials and the alcove walk model that I did not follow.