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Kazhdan - Lusztig cells and Calogero - Moser space

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Primitive ideals

$\mathfrak{g}$ - semi-simple Lie algebra / $\mathbb{C}$

$\mathfrak{h}$ - Cartan subalgebra

W - Weyl group

Prim $U(\mathfrak{g})$ - annihilators of simple $U(\mathfrak{g})$-modules

$$\lambda \longrightarrow \mathrm{Ann}\, L(\lambda)$$

order : inclusion

$$\begin{array}{ccc} \mathfrak{h}^* & \xrightarrow{\varphi} & \mathrm{Prim}\, U(\mathfrak{g}) \\ \downarrow & & \downarrow \psi \\ \mathfrak{h}^*/W\cdot & \xrightarrow[H.C]{\sim} & \mathrm{Specm}\, Z(U(\mathfrak{g})) \end{array} \qquad \begin{array}{c} I \\ \downarrow \\ I \cap Z(U(\mathfrak{g})) \end{array}$$

Dixmier, Duflo, Borho, Joseph, Jantzen, Barbasch, Vogan

Problem Understand fibres of φ, ψ

Reduces to trivial central character

$$I_w := \mathrm{Ann}\, L(w\cdot 0)$$

Def w, w' are in the same left cell if $I_w = I_{w'}$

Kazhdan - Lusztig polynomials $\rightsquigarrow$ determination of left cells (K-L conjectures)

(also combinatorial description of order on left cells)

Calogero - Moser spaces

$n \geq 2$

$$CM_n = \left\{ (M, M') \in M_n(\mathbb{C}) \times M_n(\mathbb{C}) \mid \mathrm{rk}\,([M,M'] + i\,\mathrm{id}) = 1 \right\} \Big/ GL_n(\mathbb{C})$$

$$\begin{array}{ccc} CM_n & & (n, n') \\ \pi \downarrow & & \downarrow \\ \mathbb{C}^n/S^n \times \mathbb{C}^n/S^n & & (\mathrm{spec}(n), \mathrm{spec}(n')) \end{array}$$

π finite map, degree $n!$

Kazhdan - Kostant - Sternberg, completed phase space for systems of n particles on line, interaction in $-\frac{1}{x^2}$

CM_n - smooth

Question Galois group of π?

Rational Cherednik algebras (Etingof - Ginzburg)

$$\mathbb{H} = \mathbb{C}[c] \otimes T(\mathfrak{h} \oplus \mathfrak{h}^*) \rtimes W \,/\, \text{relations}$$

relations : $[x, x'] = [\xi, \xi'] = 0$ if $x, x' \in \mathfrak{h}$, $\xi, \xi' \in \mathfrak{h}^*$

$$[\xi, x] = c \cdot \sum_{\alpha \in \Phi} \langle \alpha, x \rangle \cdot \langle \xi, \alpha^\vee \rangle \cdot s_\alpha$$

set of roots

$\mathbb{H}$ - has nice PBW type properties

Iso of vector spaces:
$$\begin{tikzcd} \mathbb{C}[c]\otimes S(\mathfrak{h})\otimes \mathbb{C}[W]\otimes S(\mathfrak{h}^*) \arrow[d, "\wr"] \\ \mathbb{H} \end{tikzcd}$$

$Q = Z(\mathbb{H})$ $\quad$ $\mathcal{Q} = \operatorname{spec} Q :=$ Calogero-Moser space associated w/ W

$\cup$

$P = S(\mathfrak{h})^W \otimes S(\mathfrak{h}^*)^W \otimes \mathbb{C}[c]$

$$\begin{tikzcd} \mathcal{Q} \arrow[d, "\pi"] & \arrow[l] \text{finite, deg } |W| \\ \mathfrak{h}^*/W \times \mathfrak{h}/W \times \mathbb{C} & \end{tikzcd}$$

<u>$c=0$</u>

$$\mathbb{H}_{c=0} = S(\mathfrak{h}\oplus\mathfrak{h}^*)\rtimes W$$
$$Q_{c=0} = S(\mathfrak{h}\oplus\mathfrak{h}^*)^W$$

<u>example</u> $\quad W = S_n \quad \mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$

$$\mathcal{Q}_{c\neq 0} \simeq CM_n$$

Frac

$$\begin{tikzcd} L \arrow[d, dash, "H"'] \\ \mathrm{Frac}(Q) \arrow[d, dash] \\ \mathbb{C}(\mathfrak{h})^W \otimes \mathbb{C}(\mathfrak{h}^*)^W \otimes \mathbb{C}(c) \arrow[uu, bend left=60, dash, "G"] \end{tikzcd}$$

Galois closure

$$|G/H| = |W|$$

$R =$ normal closure of Q in L

$\mathcal{R} = \operatorname{spec} R$

$$\begin{tikzcd}
G \curvearrowright \mathcal{R} \arrow[d] & (\mathfrak{h}\times\mathfrak{h}^*)/\Delta Z(W) \arrow[l, hook', "i"'] \arrow[d] \\
\mathcal{R}^H = \mathcal{Q} \arrow[d] & (\mathfrak{h}\times\mathfrak{h}^*)/\Delta W \arrow[l, hook'] \arrow[d] \\
\mathcal{R}^G = \mathfrak{h}^*/W \times \mathfrak{h}/W \times \mathbb{C} \arrow[d] & \mathfrak{h}/W \times \mathfrak{h}^*/W \arrow[l, hook'] \arrow[d] \\
\mathbb{C} & 0 \arrow[l, hook']
\end{tikzcd}$$

Galois closure

i - corresponds to the choice of an irreducible component of $\mathcal{R}_{c=0}$ = prime ideal r_0 of R

$$D(r_0) := \mathrm{Stab}_G(r_0) \xrightarrow[\exists]{\sim} (W\times W)/\Delta Z(W)$$

$$D(r_0)\cap H \xrightarrow{\sim} \Delta(W/Z(W)) \quad (\subset (W\times W)/\Delta Z(W))$$

$G = D(r_0)\cdot H$

$G/H \xrightarrow{\sim} D(r_0)/D(r_0)\cap H \xrightarrow{\sim} (W\times W)/\Delta W \xleftarrow{\sim} W$

$(w,1) \longleftarrow\!\shortmid\; w$

$G \hookrightarrow \mathrm{Sym}(G/H) = \mathrm{Sym}(W)$

r prime ideal of R above $\{0\}\times\mathfrak{h}/W\times\{1\}$

$$\begin{array}{ccccc}
I(r) & \subset & D(r) & \subset & G \\
\| & & \| & & \\
\mathrm{Fix}_G\,\mathbb{C}(r) & & \mathrm{Stab}_G(r) & &
\end{array}$$

<u>Conjecture</u> left cells = $I(v)$-orbits in W.

true for types A_2 and B_2

for type B_2 G = weyl group type $D_4 \subset S_8$

<u>Remarks</u>

right et cells, two sided cells should arise as well

"parabolic" cells

Missing: $\mathbb{R}_{>0}$-action, would give inclusion between primitive ideals

going back to $\mathbb{H}$ one can construct reps of $W \overset{?}{=}$ left cell reps.

Rep. of Cherednik algebras (version w/ extra parameter t)

$gr\, \mathbb{H}_t \simeq S(\mathfrak{h} \oplus \mathfrak{h}^*) \rtimes W$ simple mod for $\mathbb{H}_{t=1}$ in cat $\mathcal{O}$

$\cup$

$\mathbb{C}[(\mathfrak{h} * \mathfrak{h}^*)/W]$ char cycle in $(\mathfrak{h} \times \mathfrak{h}^*)/W$

optimistic hopes:

$\mathcal{O}$ for $\mathfrak{g}$ can be recovered from calogero-moser spaces

$\mathcal{O} \overset{?}{\longleftrightarrow}$ wrapped Fukaya category of $\mathcal{C}$.