Geometric Langlands and physics

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Q1: Is either of the two sides of the Geometric Langlands correspondence related to physics?

Q2: Is there a duality (equivalence) of the correspondin-g physical theories?

Q3 Does the physics make anything obvious?

Two approaches

1) 2-dim CFT (Frenkel)

2) various TFTs (Kapustin, Witten)

will mainly discuss 2) in this talk.

G - compact Lie group

$G_{\mathbb{C}}$ - complexification

C - connected oriented 2-dim $\mathbb{R}$-manifold.

w/o boundary

$\mathcal{Y}(G,C)$ = moduli space of flat connections on $G_{\mathbb{C}}$-bundles $E_{\mathbb{C}} \to C$ / gauge transformations

$\simeq \mathrm{Hom}(\pi_1(C), G_{\mathbb{C}})/\sim$

$\mathcal{Y}(G,C)$ is a holomorphic symplectic manifold.

The complex structure comes from the complex structure on $G_{\mathbb{C}}$.

$\mathrm{ad}\, E$ = adjoint bundle = $E \times^{G} \mathfrak{g}$

The intersection pairing on $H^1(C, \mathrm{ad}\,E)$

$$\alpha_1, \alpha_2 \in \Omega^{(0,1)}(\mathrm{ad}\,E), \quad \beta_1, \beta_2 \in \Omega^{(1,0)}(\mathrm{ad}\,E)$$

$$(\alpha, \beta) = \int_C \mathrm{Tr}(\alpha_1 \beta_2 - \alpha_2 \beta_1)$$

induces a symplectic form on $\mathcal{Y}(G, C)$.

Holomorphic symplectic manifold $\Rightarrow$ trivial canonical bundle (i.e., Calabi-Yau)

Every CY-manifold is equipped w/ a symplectic form ω and a holomorphic n-form Ω, $n = \dim X$, nowhere vanishing.

To any CY-manifold we can associate two TFTs: A-model, B-model

A-model $\uparrow$ view X as a real symplectic manifold

B-model $\uparrow$ view X as a complex manifold

$A(X)$: physical observables

$$H^*_A = \bigoplus_{p,q} H^q(X, \Lambda^p T^* X)$$

$$= \bigoplus_{p,q} H^{p,q}(X)$$

- correlation functions depend only on symplectic structure

- maps $f : \Sigma \longrightarrow X$, ms f is holomorphic

Σ: Riemann surface

$$B(X): \quad H^*_B = \bigoplus^q H^q(X, \Lambda^p T_X) \simeq \bigoplus_{p,q} H^{n-p,q}(X).$$

- correlation functions depend only on complex structure
- f has to be constant

<u>Mirror symmetry</u>: Calabi-Yau manifolds come in pairs $(X, X^\vee)$ such that $A(X) = B(X^\vee)$, $A(X^\vee) = B(X)$.

<u>Consequences</u>:

- classical mirror symmetry: for a mirror pair $(X, X^\vee)$ we have
$$h^{p,q}(X) = h^{n-p,q}(X^\vee).$$
- Strominger-Yau-Zaslow conjecture ('96)
  1) Every CY-manifold of $\mathbb{R}$-dim $2n$ admits a fibration $\pi: X \longrightarrow B$ s.t $\pi^{-1}(b) \simeq T^n$, is special Lagrangian, i.e., $\omega|_{\pi^{-1}(b)} = 0$; $\operatorname{Im}\Omega|_{\pi^{-1}(b)} = 0$ $\iff \operatorname{Re}\Omega|_{\pi^{-1}(b)} \propto \operatorname{Vol}(\pi^{-1}(b))$ for $b \in B \setminus \Delta$, codim $\Delta \leq 2$
  2) CY manifolds come in pairs $(X, X^\vee)$ s.t. $f: X \longrightarrow B$, $f^\vee: X^\vee \longrightarrow B^\vee$ satisfy $B = B^\vee$, $\Delta \neq \Delta^\vee$
$$(f^\vee)^{-1}(b) = \operatorname{Hom}(f^{-1}(b), S^1)$$

Homological mirror symmetry (Kontsevich '94)

To the A and B model one can associate A_∞-catego-ries $A(X)$, $B(X)$

~~A(X)~~ $B(X) = D^b(\mathrm{Coh}(X))$

$A(X) = D(\mathrm{Fuk}(X))$

$A(X)$ is A_∞-equivalent to $B(X^\vee)$ and $A(X^\vee)$ is A_∞-equivalent to $B(X)$.

Geometric Langlands correspondence

$$\left\{ \begin{matrix} \text{flat } {}^LG \text{ bundles} \\ \text{on } C \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{Hecke eigensheaves} \\ \text{on } \mathrm{Bun}_G(C) \end{matrix} \right\}$$

What is ${}^LG$?

$T \subset G$ maximal torus

$X^*(T) = \mathrm{Hom}(T, U(1))$ weight lattice
$\quad \cup$
$\quad \Delta$ roots

$X_*(T) = \mathrm{Hom}(U(1), T)$ coweight lattice
$\quad \cup$
$\quad \Delta^\vee$ co-roots

Root data of G: $(X^*(T), X_*(T), \Delta, \Delta^\vee)$

${}^LG$ given by root data $(X_*(T), X^*(T), \Delta^\vee, \Delta)$

examples

G	${}^LG$
GL_n	GL_n
SL_n	PGL_n
Sp_{2n}	$SO(2n+1)$
$Spin_{2n}$	$SO(2n)/\mathbb{Z}_2$
E_8	E_8

$\mathcal{Y}(G, C)$ and $\mathcal{Y}({}^L G, C)$, choose a complex structure on C.

consider $E \longrightarrow C$ G-bundle (not $G_{\mathbb{C}}$)

A - connection on E

$\phi \in \Omega^1(C, \mathrm{ad}\, E)$

Hitchin's equations ('87):

$$\left.\begin{aligned} F - \phi \wedge \phi &= 0 \\ D\phi = D{*}\phi &= 0 \end{aligned}\right\} (*) \quad (F \text{ curvature 2-form of } A)$$

solutions have two interpretations

1) $\mathcal{A} = A + i\phi \quad \Rightarrow \mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0$

$\Rightarrow$ a solution of $(*)$ defines a point in $\mathcal{Y}(G, C)$.

2) $A^{(0,1)}$ defines a complex structure on $E_{\mathbb{C}}$

$\Rightarrow E_{\mathbb{C}}$ is a holomorphic $G_{\mathbb{C}}$-bundle on C

$\phi = \varphi + \bar{\varphi}$, $\varphi = (1,0)$ part of ϕ

$(*) \Rightarrow \varphi \in H^1(C, K_C \otimes \mathrm{ad}\, E)$

The pair (E, φ) is called a Higgs-bundle.

$\mathcal{M}_H(G)$ = moduli space of solutions to Hitchin's equations / gauge equivalences

Recall: A hyperkähler metric on a $4n$-dim real manifold is a Riemannian metric g which is

Kähler w.r.t three complex structures I, J, K satisfying the relations of the quaternions

$I^2 = J^2 = K^2 = -1$, $IJ = -JI = K$ + cyclic permutations

Define: $\omega_I(X,Y) = g(IX, Y)$ (and ω_J, ω_K)

$\Omega_I(X,Y) = \omega_J + i\omega_K$ (and Ω_J, Ω_K)

<u>Thm</u> (Hitchin '87)

$\mathcal{M}_H(G)$ is a hyperkähler manifold. Moreover w.r.t I $\mathcal{M}_H(G)$ parametrizes semistable Higgs bundles (E, φ)

w.r.t J $\mathcal{M}_H(G) = \mathcal{Y}(G, C)$

The natural holomorphic symplectic form Ω on $\mathcal{Y}(G,C)$ is identified w/ $i\Omega_J$. The real symplectic form ω on $\mathcal{Y}(G,C)$ is identified w/ Ω_K.

J, Ω_J <u>do not</u> depend on the choice of complex structure on C.

P - invariant polynomial of degree d on $\mathfrak{g}$

$$P(\varphi) \in H^0(C, K_C^{\otimes d})$$

Define: $\pi : \mathcal{M}_H(G) \longrightarrow \bigoplus_{i=1}^{\text{rank of } \mathfrak{g}} H^0(C, K_C^{\otimes d_i}) = B$

$$(E, \varphi) \longmapsto (P_1(\varphi), \ldots, P_{\text{rank of } \mathfrak{g}}(\varphi))$$

Thm (Hitchin '87) π is a fibration by (torsors of) complex abelian varieties in complex structures I, $\pi^{-1}(b)$ are Lagrangian wrt Ω_I.

Cor (Hausel, Thaddeus '02) $\pi^{-1}(b)$ are Lagrangian for ω_K, holom. w.r.t I $\Rightarrow$ $\pi^{-1}(b)$ are area minimi-zing in their homology class, i.e. special Lagrangian.

(π is called the Hitchin fibration)

Thm (Hausel, Thaddeus '02) $G = SU_n$

$\mathcal{M}_H(G)$ is mirror to $\mathcal{M}_H({}^LG)$ in the sense that $\mathcal{M}_H(G) \xrightarrow{\pi} B \xleftarrow{{}^L\pi} \mathcal{M}_H({}^LG)$ form an SYZ mirror pair of special Lagrangian fibrations s.t.

$$\dim H^{*,*}(\mathcal{M}_H(G)) = \dim H^{n-*,*}(\mathcal{M}_H({}^LG))$$

$F_b = \pi^{-1}(b)$ ${}^LF_b = {}^L\pi^{-1}(b)$ are dual in the sense that

$$\{\text{pts in } {}^LF_b\} \xleftrightarrow{1-1} \{\text{flat holom. line bundle on } F_b\}$$

In other words: the B-model of $\mathcal{M}_H({}^LG)$ in complex structure I is mirror to the A-model of $\mathcal{M}_H(G)$ w/ symplectic structure ω_K.

homological mirror symmetry $A(\mathcal{M}_H({}^LG)) = B(\mathcal{M}_H(G))$