Geometric Langlands correspondence for $GL_1(\mathbb{C})$ ①

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Introduction

Thm (Deligne [$n=1$], Drinfeld [$n=2$], Lafforgue [$n>2$]) (Geometric Langlands correspondence for $GL_n(\mathbb{C})$)

X - smooth projective curve / $\mathbb{C}$

for ~~all~~ each ~~the~~ irred. local systems $\mathcal{L}$ on X, there exists an 'automorphic' sheaf $Aut_E \in \mathcal{D}\text{-mod}(Bun_n(X))$

s.t Aut_E is irred + is a Hecke eigensheaf.

$$\begin{tikzcd} & Hecke_n \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] & \\ Bun_n(X) & & Bun_n(X) \times X \end{tikzcd}$$

$$\overrightarrow{h}_* \overleftarrow{h}^* Aut_E = Aut_E \boxtimes \Lambda^i E \quad [\text{some shift}]$$

$$Pic(X) = \bigsqcup_d Pic^d(X) = \{\text{divisors of deg. } d\}/\sim$$

$\mathcal{L} \in Pic(X)$, $p \in X$ $\quad \mathcal{L}(p) \simeq \mathcal{L} \otimes \mathcal{O}(p)$

Thm for ~~all~~ each irred. local system $E \in Loc^1(X)$, $\exists$ $Aut_E \in Loc^1(Pic(X))$ s.t

1) Aut_E is irred

2)
$$\begin{tikzcd} & Pic(X) \times X \arrow[dl, "\overleftarrow{h}"', "\mathcal{L} \mapsto \mathcal{L} \otimes \mathcal{O}_X(x)"] \\ Pic(X) & \end{tikzcd}$$
$\overleftarrow{h}^* Aut_E \simeq Aut_E \boxtimes E$

Example if genus $(X) = 0$ $\left(\Rightarrow \pi_1(X) = 0 \right)$ $\underset{\mathbb{P}^1}{\wr\wr}$

so unique irred. local system on X is $\mathbb{C}_X$

$\mathrm{Pic}^0(X) = \mathbb{Z}$ $\rightsquigarrow$ Thm is trivially true.

proof of thm 2

Def: Abel-Jacobi map

$S^d X = X^d / S_d = \{$ set of all effective divisors of degree $d \}$

$$\pi_d : \quad S^d X \longrightarrow \mathrm{Pic}^d(X)$$
$$[x_1, \dots, x_d] \longmapsto \mathcal{O}(\Sigma x)$$

3 steps

Step 1 prove analogue of Thm 2 for

$$\begin{array}{ccc} S^d X \times X & \longrightarrow & S^{d+1} X \\ {\scriptstyle \pi_d \times id} \downarrow & & \downarrow {\scriptstyle \pi_{d+1}} \\ \mathrm{Pic}^d(X) \times X & \xrightarrow{\vec{h}} & \mathrm{Pic}^{d+1}(X) \end{array}$$

Step 2 Descend to $\mathrm{Pic}^d(X)$ via π_d

Step 3 Extend to all of Pic

Step 1

Prop

$$\begin{array}{ccc} q : X^d & \longrightarrow & S^d X \\ g \cdot \downarrow & \nearrow & \\ X^d & & \end{array}$$

quotient map

$E^{(d)} = q_*(E^{\boxtimes d})^{S_d}$ is an irred. loc system (rank

$\mathcal{F} \in Loc(S^d X) \qquad q^*(\mathcal{F}) \simeq E^{\boxtimes d} \iff \mathcal{F} \simeq E^{(d)}$

$\overleftarrow{h} : S^d X \times X \to S^{d+1} X$

$([x_1, \dots, x_d], x_{d+1}) \longmapsto [x_1, \dots, x_{d+1}]$

$\overleftarrow{h}^* E^{(d+1)} \simeq E^{(d)} \boxtimes E$

proof

1) $E^{(d)}\big|_{[x_1,\dots,x_d]} = E^{\boxtimes d}\left(q^{-1}([x_1 \dots x_d])\right)^{S_d} \simeq \Big(\bigoplus_{\substack{(x_1,\dots,x_d) \\ \uparrow \\ [x_1,\dots,x_d]}} \mathbb{C}\Big)^{S_d} \simeq \mathbb{C}$

2) $q^*(E^{(d)}) \xrightarrow{\text{incl}} q^* q_* E^{\boxtimes d} \xrightarrow{\text{adj}} E^{\boxtimes d}$

if $q^* F \simeq E^{\boxtimes d}$

3)
$$\begin{array}{ccc} S^d X \times X & \longrightarrow & S^{d+1} X \\ \uparrow & & \\ X^d \times X & \xrightarrow[\text{id}]{} & X^d \times X \end{array}$$

$\overleftarrow{h}^* E^{(d)} \simeq E^{(d)} \boxtimes E$

Step 2

fact $\quad \pi_d : S^d X \longrightarrow Pic^d(X)$

1) if $d > 2g-2$

π_d is surjective (Jacobi inversion)

2) $d > 2g-2$

$\pi_d^{-1}([D]) = \{\text{effective divisors} \sim D\}$

$\simeq \mathbb{P} H^0(X, \mathcal{O}_X(D))$

Riemann-Roch : $\dim \mathbb{P} H^0(X, \mathcal{O}_X(D)) + 1 - \dim H^0(X, K-D) = d - g + 4$

Lemma π_d is a d-g dimensional projective bundle ④

(I think I missed a hypothesis for this lemma)

$d > 2g-2$

$$\begin{tikzcd} \mathbb{CP}^{d-g} \arrow[r] & S^d X \arrow[d, "\pi_d"] \\ & \mathrm{Pic}^d(X) \end{tikzcd}$$

Def: Define Aut_E^d by $\pi_d^*(Aut_E^d) \simeq E^{(d)}$ for $d > 2g-2$.

<u>Step 3</u>

Aut_E^d for $d > 2g-2$

$\overleftarrow{h} : \mathrm{Pic}^d(X) \longrightarrow \mathrm{Pic}^{d+1}(X)$, $\mathcal{L} \longmapsto \mathcal{L}(x)$

if $d > 2g-1$

$$\begin{tikzcd} \mathrm{Pic}^{d-1}(X) \times X \arrow[dr, "\overleftarrow{h}"] & \\ \mathrm{Pic}^{d-1}(X) \arrow[u, hook, "i_x"'] \arrow[r, "h_x"] & \mathrm{Pic}^d(X) \end{tikzcd}$$

$$\overleftarrow{h}_x^*(Aut_E^d) \simeq i_x^*(Aut_E^{d-1} \boxtimes E$$

$$\simeq Aut_E^{d-1} \otimes E_x \quad \cancel{\simeq Aut_E^{d-1}}$$

<u>Def</u> Assume Aut_E^d is defined for ~~some $k \in \mathbb{Z}$~~

all ~~n~~ $d \geq k$, for some $k \in \mathbb{Z}$

$$Aut_E^{k-d} = \overleftarrow{h}_*(Aut_E^k) \otimes E_x^\vee$$

<u>Prop</u> Aut_E^k defined this way is a Hecke eigensheaf.