Cotangent complexes of moduli spaces and symplectic structures (Z. Zhang) (Freiburg, 20 Jul 2012) ①

Background

X - complex K3 surface

H - ample line bundle

$v \in H^{even}(X, \mathbb{Z})$ Mukai vector ($\Leftrightarrow$ rank + chern classes)

Thm (Maruyama, Gieseker, Simpson, ...)

$M_{X,H}(v)$ moduli space of semistable sheaves w/ Mukai vector v

1) projective scheme

2) parameterizes S-equivalent classes of sheaves w/ Mukai vector v

Def F semistable w.r.t H if for

$$\forall\, 0 \neq E \subsetneq F, \quad \frac{\chi(E(m(H)))}{rk\,E} \leq \frac{\chi(F(mH))}{rk\,F} \quad \text{for } m \gg 0$$

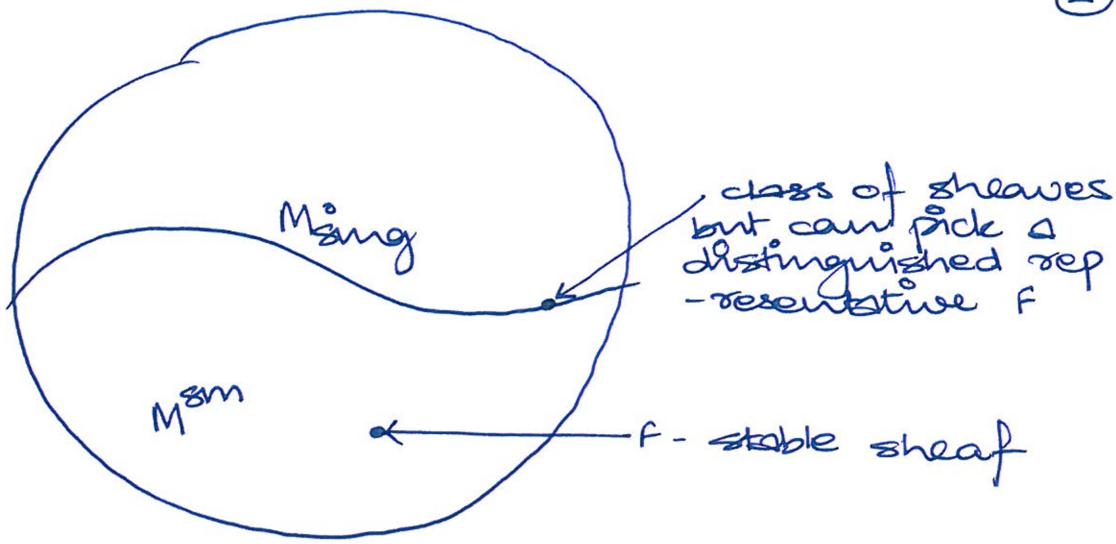
extra assumptions

1) H is generic

2) $M^{st} \neq \phi$

Fact

$M_{X,H}(v)$ is $\begin{cases} \text{smooth} & \text{if } v \text{ is primitive} \\ \text{singular} & \text{otherwise} \end{cases}$



<u>smooth case</u>

<u>Thm</u> (Mukai) when v is primitive,

$M_{X,H}(v)$ is an <u>irreducible</u> <u>holomorphic</u> <u>symplectic</u> <u>manifold</u> (IHSM)

IHSM = Kähler + $\pi_1 = 0$ + $h^{2,0} = 1$
+ symplectic form $\sigma \in H^0(\Omega^2)$

$\sigma : TM \times TM \longrightarrow \mathcal{O}_M$
anti-symmetric + non-degenerate
i.e., $\theta : \Omega_M \xrightarrow{\sim} \Omega^\vee_M$, $\theta^\vee = -\theta$

<u>singular case</u>

<u>Q1</u> explicit description of singularities

<u>Q2</u> realize them as symplectic objects.

<u>lory A</u> an analytic open neighborhood at $[F] \in M$

$\mu^{-1}(0) /\!/ \mathrm{Aut}(F)$

where $\mu : \mathrm{Ext}^1(F,F) \longrightarrow \mathrm{Ext}^2(F,F)$ (homog. quadratic)
$e \longmapsto e \cup e$

<u>Conj B</u> The DG-algebra $R\mathrm{Hom}^\bullet(F,F)$ is formal i.e., quasi-isomorphic to $\mathrm{Ext}^\bullet(F,F)$

<u>Conj B</u> $\Longrightarrow$ <u>Conj A</u>

<u>Thm</u> Conj B is true when
(Kaledin-Lehn) : F_i are identical of rk 1
(Z.) : either F_i's are all of rk 1
or F_i's are all of rank ≥ 2

<u>symplectic</u> <u>structure</u> ?

previous attempts : 1) symplectic resolution (O'Grady, Kaledin-Lehn-Sorger)

2) symplectic singularities (Beauville, ...)

3) symplectic structure on derived schemes (Pantev-Töen-Vaquié-Vezzosi)

<u>new idea</u>

moduli scheme M $\rightsquigarrow$ moduli stack $\mathcal{M}$

cotangent bundle Ω_M $\rightsquigarrow$ cotangent complex $\mathbb{L}_{\mathcal{M}}$

<u>Hope</u> $\exists\ \theta : \mathbb{L}_{\mathcal{M}} \xrightarrow{\sim} \mathbb{L}_{\mathcal{M}}^\vee$ such that $\theta^\vee = -\theta$

construction

$M = Q /\!/ G$ GIT - quotient

Q = open subscheme of Quot

$G = PGL_n$

$\mathcal{M} = [Q/G]$ global quotient

notation universal quotient on $Q \times X$

$0 \to K \to \mathcal{E} \to \mathcal{F} \to 0$

all flat over Q

$\mathcal{E}$ trivial over Q

$\pi : Q \times X \longrightarrow Q$ projection

$q : Q \longrightarrow \mathcal{M} = [Q/G]$ quotient

Thm (Z.)

1) $q^* \mathbb{L}_{\mathcal{M}} = R\pi_* R\mathcal{H}om(\mathcal{F}, \mathcal{F})^\vee_0 [-1]$

$\in \mathrm{Perf}_Q [-1, 1]$

2) symplectic

$\exists\, \theta : q^* \mathbb{L}_{\mathcal{M}} \xrightarrow{\sim} (q^* \mathbb{L}_{\mathcal{M}})^\vee$ ~~quasi~~ qis

s.t. $\theta^\vee = -\theta$

3) θ is invariant under G-action