ORDINARY (CO)HOMOLOGY

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Contents

1. Categories and functors

1.1. Algebraic topology concerns mappings from topology to algebra. The appropriate language is that of categories and functors.

1.2. A category $\mathcal{C}$ consists of a collection of objects, a set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ of *morphisms* for any two objects $A, B \in \mathcal{C}$, an *identity morphism* $\mathrm{id}_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ for each object $A \in \mathcal{C}$ (usually abbreviated to id), and a composition law

$$\circ\colon \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$$

for each triple of objects $A, B, C \in \mathcal{C}$. Composition is required to be associative and identity morphisms are required to behave as their name indicates:

$$f \circ (g \circ h) = (f \circ g) \circ h, \quad f \circ \mathrm{id} = f \quad \text{and} \quad \mathrm{id} \circ f = f$$

whenever the specified compositions make sense.

1.3. *Remark.* The subscript '$\mathcal{C}$' is often omitted from '$\mathrm{Hom}_{\mathcal{C}}$' if there is no chance of confusion (and by some authors even if there is). Further, the composition $f \circ g$ is often just written fg.

1.4. A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is called an *isomorphism* if there exists a morphism $g \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ such that $fg = \mathrm{id}$ and $gf = \mathrm{id}$.

1.5. **Example. Set** is the category of sets with morphisms given by maps of sets. Isomorphisms in this category are bijective maps.

1.6. **Example. Ab** is the category of abelian groups with morphisms given by group homomorphisms. Isomorphisms in this category are group homomorphisms that are injective and surjective as maps of the underlying sets.

1.7. A *functor* $F\colon \mathcal{C} \to \mathcal{D}$ between two categories $\mathcal{C}$ and $\mathcal{D}$ is a rule that assigns to each object $A \in \mathcal{C}$ an object $F(A) \in \mathcal{D}$, to each morphism $f\colon A \to B$ in $\mathcal{C}$ a morphism $F(f)\colon F(A) \to F(B)$ in $\mathcal{D}$ in such a way that

$$F(\mathrm{id}_A) = \mathrm{id}_{F(A)} \quad \text{and} \quad F(f \circ g) = F(f) \circ F(g).$$

More precisely, this is a *covariant* functor. A *contravariant* functor reverses the direction of morphisms, so that F sends $f\colon A \to B$ to $F(f)\colon F(B) \to F(A)$ and satisfies $F(f \circ g) = F(g) \circ F(f)$.

1.8. A *natural transformation* $\alpha\colon F \to G$ between functors $F, G\colon \mathcal{C} \to \mathcal{D}$ is the data of a morphism $\alpha_A\colon F(A) \to G(A)$, for each $A \in \mathcal{C}$, such that the following diagram commutes for each morphism $f\colon A \to B$ in $\mathcal{C}$:

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow[\alpha_B]{} & G(B) \end{array}$$

The notion of *natural isomorphism* is formulated in the obvious way. The functors F and G are said to be equivalent if there exists a natural isomorphism $F \xrightarrow{\sim} G$. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is said to be an *equivalence of categories* if there exists a functor $G\colon \mathcal{D} \to \mathcal{C}$ such that $FG\colon \mathcal{D} \to \mathcal{D}$ and $GF\colon \mathcal{C} \to \mathcal{C}$ are equivalent to the identity functor on $\mathcal{D}$ and $\mathcal{C}$ respectively.

1.9. **Example.** Let $\mathbf{Vect}_k$ be the category of vector spaces over some field k with morphisms being linear transformations. Define a contravariant functor ${}^*\colon \mathbf{Vect}_k \to \mathbf{Vect}_k$ by assigning to a vector space V its linear dual V^* and assigning to a linear map $f\colon V \to W$ the map $f^*\colon W^* \to V^*$ given by $(f^*\phi)(v) = \phi(f(v))$, $\phi \in W^*, v \in V$. Then we obtain a natural transformation from the identity functor on $\mathbf{Vect}_k$ to ** by mapping a vector $v \in V$ to $\mathrm{eval}_v \in V^{**}$, where eval_v is defined by $\mathrm{eval}_v(\phi) = \phi(v)$, $\phi \in V^*$. If instead of all vector spaces we consider the category of finite-dimensional vector spaces, then this natural transformation is an isomorphism and this functor of 'taking the dual' is an equivalence.

1.10. *Remark.* The terms 'natural', 'functorial' and 'canonical' will be used interchangeably with 'a natural transformation of functors'. Thanks to the laziness of the author, often the functors in question will not be explicitly specified, they will be 'obvious' from the context.

1.11. **Exercise.** *Open your favorite linear algebra textbook, find all instances of the terms 'natural' and 'canonical'. Reconcile the usage of these terms with the remark above.*

1.12. **Exercise.** *Open your favorite (or least favorite) textbook on algebraic (or differential) geometry. Find all instances of the terms 'natural' and 'canonical'. Reconcile the usage of these terms with the remark above.*

2. Exact sequences

2.1. In this section 'module' = 'module over some fixed ring' and 'map' = 'module homomorphism'. Not much will be lost if the reader equates 'module' with 'abelian group' and 'map' with 'group homomorphism'.

2.2. A sequence of modules and maps

$$\cdots \xrightarrow{f_{i-1}} M^i \xrightarrow{f_i} M^{i+1} \xrightarrow{f_{i+1}} \cdots$$

is said to be *exact* at M^i if $\mathrm{im}(f_{i-1}) = \ker(f_i)$. The sequence is exact if it is exact at each M^i.

2.3. **Example.** $0 \to M' \xrightarrow{f} M$ is exact if and only if f is injective.

2.4. **Example.** $M \xrightarrow{g} M'' \to 0$ is exact if and only if g is surjective.

2.5. **Example.** A sequence

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

is exact if and only if f is injective, g is surjective and g induces an isomorphism of $\mathrm{coker}(f) = M/\mathrm{im}(f)$ onto M''. Such an exact sequence is called a *short exact sequence.*

3. Conventions and notation regarding topological spaces

3.1. A space will always mean a topological space and a map between spaces will always mean a continuous map. The category of spaces and maps will be denoted **Top**.

3.2. The real numbers will be denoted by $\mathbf{R}$ and the complex numbers by $\mathbf{C}$. Some standard spaces that we will use are:

- *The one point space*: pt consisting of a single point.
- *Euclidean n-space*: $\mathbf{R}^n$ with the usual metric.
- *The n-disk*: $D^n = \{(x_1, \dots, x_n) \in \mathbf{R}^n \mid x_1^2 + \cdots + x_n^2 \leq 1\}$.
- *The $n-1$-sphere*: $S^{n-1} = \{(x_1, \dots, x_n) \in \mathbf{R}^n \mid x_1^2 + \cdots + x_n^2 = 1\}$. We will also need the following alternate models for spheres: S^{n-1} is the boundary ∂D^n of the n-disk in $\mathbf{R}^n$; S^{n+1} is the quotient $D^n/\partial D^n$; the 1-sphere (or circle) is $S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$.
- *The unit interval*: $I = [0,1] \subset \mathbf{R}$. Note that $I = D^1$.

3.3. Let $i\colon A \hookrightarrow X$ be the inclusion of a subspace. A *retraction* of X to A is a map $r\colon X \to A$ such that $ri = \mathrm{id}$. If such a retraction exists, then we say that X retracts to A.

3.4. **Example.** Let $x \in X$. Then X retracts to x.

3.5. A *homotopy* is a map $h\colon X \times I \to Y$. Write h_0 for the map $X \to Y$, $x \mapsto h(x,0)$ and h_1 for the map $X \to Y$, $x \mapsto h(x,1)$. Then we say that h is a homotopy between h_0 and h_1. A map $f\colon X \to Y$ is said to be homotopic to $g\colon X \to Y$ if there exists a homotopy $h\colon X \times I \to Y$ such that $f = h_0$ and $g = h_1$. This is an equivalence relation on maps $X \to Y$. We will write **hTop** for the category with objects spaces and morphisms homotopy equivalence classes of maps.

3.6. Two spaces are *homotopy equivalent* if they are isomorphic in **hTop**. That is, X is homotopy equivalent to Y if there exist maps $f\colon X \to Y$ and $g\colon Y \to X$ such that fg and gf are homotopic to the identity on Y and X respectively. We say that f and g are homotopy equivalences. If a space is homotopy equivalent to pt, then it is called *contractible.*

3.7. Let $i\colon A \hookrightarrow X$ be the inclusion of a subspace. A *deformation retract* from X to A is a retraction $r\colon X \to A$ such that ir is homotopy equivalent the identity on X. A deformation retract is a homotopy equivalence.

3.8. Let X be a space. A *path* from a point $x \in X$ to a point $y \in X$ is a map $f\colon [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$. This defines an equivalence relation on points of X. The corresponding equivalence classes are called the *path components* of X. The path components are precisely the homotopy equivalence classes of maps pt $\to X$. For $x \in X$ we write $[x]$ for the corresponding path component.

3.9. The set of path components of a space X, denoted $\pi_0(X)$, is the most intuitively obvious invariant of X. In fact, it is a homotopy invariant: if X is homotopy equivalent to Y, then $\pi_0(X) \simeq \pi_0(Y)$. The fundamental group $\pi_1(X)$, and more generally the higher homotopy groups $\pi_n(X)$, are higher dimensional analogues of $\pi_0(X)$. Unfortunately, homotopy groups can be extremely hard to compute (to this day we do not have complete information about the homotopy groups of spheres). Homology and cohomology groups are a different higher dimensional analogue of $\pi_0(X)$. They are harder to define than homotopy groups, but are comparatively easier to compute.

4. (Co)homology

4.1. Let $H_0(X)$ be the free abelian group on the set of path components of X. Given a map $f\colon X \to Y$ define $H_0(f)\colon H_0(X) \to H_0(Y)$ by extending the assignment $[x] \mapsto [f(x)]$ on path components linearly. By a standard abuse of notation, we write f_* instead of $H_0(f)$. The map $f_*\colon H_0(X) \to H_0(Y)$ only depends on the homotopy class of f. In other words, we have a functor

$$H_0\colon \mathbf{hTop} \to \mathbf{Ab}.$$

The group $H_0(X)$ is called the *zeroth homology group* of X.

4.2. Let $H^0(X)$ be the group of homomorphisms $H_0(X) \to \mathbf{Z}$. Given a map $f\colon X \to Y$ define $f^*\colon H^0(Y) \to H^0(X)$ by $f^*\varphi([x]) = \varphi(f_*[x])$, $\varphi \in H^0(Y)$, $[x] \in H_0(X)$. So we have a contravariant functor

$$H^0\colon \mathbf{hTop} \to \mathbf{Ab}.$$

4.3. **Exercise.** *Let $A, B \subseteq X$ be open subsets that cover X. Let $\alpha\colon A \cap B \hookrightarrow A$, $\beta\colon A \cap B \hookrightarrow B$, $a\colon A \hookrightarrow X$ and $b\colon \hookrightarrow X$ be the inclusion maps. Are either of the following sequences exact? Prove or find counterexamples.*

$$0 \to H_0(A \cap B) \xrightarrow{\left(\begin{smallmatrix}\alpha_* \\ -\beta_*\end{smallmatrix}\right)} H_0(A) \oplus H_0(B) \xrightarrow{(\,a_* \;\; b_*\,)} H_0(X) \to 0,$$

$$0 \to H^0(X) \xrightarrow{\left(\begin{smallmatrix}a^* \\ b^*\end{smallmatrix}\right)} H^0(A) \oplus H^0(B) \xrightarrow{(\,\alpha^* \;\; -\beta^*\,)} H^0(A \cap B) \to 0.$$

4.4. **Problem.** *Same question as the previous exercise but assume that X is simply connected.*

4.5. **Theorem.** *For $q \in \mathbf{Z}_{\geq 1}$, there exist (contravariant) functors $H^q\colon \mathbf{hTop} \to \mathbf{Ab}$ together with canonical maps $\delta\colon H^{q-1}(A \cap B) \to H^q(X)$, where $A, B \subseteq X$ are open subsets that cover X, such that:*

(i) *$H^q(\mathrm{pt}) = 0$ for all $n \neq 0$;*
(ii) *if $\alpha\colon A \cap B \hookrightarrow A$, $\beta\colon A \cap B \hookrightarrow B$, $a\colon A \hookrightarrow X$ and $b\colon B \hookrightarrow X$ are the inclusion maps, then the following sequence is exact for all $q \in \mathbf{Z}_{\geq 0}$:*

$$\cdots \xrightarrow{\delta} H^q(X) \xrightarrow{\left(\begin{smallmatrix}a^* \\ b^*\end{smallmatrix}\right)} H^q(A) \oplus H^q(B) \xrightarrow{(\,\alpha^* \;\; -\beta^*\,)} H^q(A \cap B) \xrightarrow{\delta} H^{q+1}(X) \to \cdots$$

(iii) *if X is the disjoint union of a set of spaces X_i, then the inclusions $X_i \hookrightarrow X$ induce an isomorphism $H^q(X) \xrightarrow{\sim} \prod_i H^q(X_i)$ for all q.*

Here, by convention we are writing f^* instead of $H^q(f)$.

Proof. Postponed. □

4.6. *Remark.* The exact sequence above is called the *Mayer-Vietoris sequence* associated to A and B.

4.7. There is an analogous statement for homology:

4.8. **Theorem.** *For $q \in \mathbf{Z}_{\geq 1}$, there exist functors $H_q \colon \mathbf{hTop} \to \mathbf{Ab}$ together with canonical maps $\partial \colon H_q(X) \to H_{q-1}(A \cap B)$, where $A, B \subseteq X$ are open subsets that cover X, such that:*

(i) *$H_q(\mathrm{pt}) = 0$ for all $q \neq 0$;*
(ii) *if $\alpha \colon A \cap B \hookrightarrow A$, $\beta \colon A \cap B \hookrightarrow B$, $a \colon A \hookrightarrow X$ and $b \colon B \hookrightarrow X$ are the inclusion maps, then the following sequence is exact for all $q \in \mathbf{Z}_{\geq 0}$*

$$\xrightarrow{\partial} H_{q+1}(A \cap B) \xrightarrow{\left(\begin{smallmatrix} \alpha_* \\ -\beta_* \end{smallmatrix}\right)} H_{q+1}(A) \oplus H_{q+1}(B) \xrightarrow{(\, a_* \;\, b_* \,)} H_{q+1}(X) \xrightarrow{\partial} \cdots$$

(iii) *if X is the disjoint union of a set of spaces X_i, then the inclusions $X_i \hookrightarrow X$ induce an isomorphism $\bigoplus_i H_q(X_i) \xrightarrow{\sim} H_q(X)$ for all q.*

Here, by convention we are writing $f_* = H(f)$ for a map f.

Proof. Postponed. □

4.9. **Exercise.** *In view of Remark 1.10, the maps δ and ∂ are natural transformations between some functors. Which functors?*

4.10. *Remark.* The formalism of cohomology and homology is very similar. However, it will eventually turn out that cohomology has additional structure that homology doesn't. So I will focus on cohomology. It is a running exercise to appropriately reformulate all statements that follow for homology.

4.11. **Example.** As $\mathbf{R}^n$ is contractible, we have

$$H^*(\mathbf{R}^n) = H^*(\mathrm{pt}).$$

4.12. **Exercise.** *Let $A \subseteq X$ be a subspace and let $i \colon A \hookrightarrow X$ be the inclusion map. Suppose A is a retract of X. Show that $i^* \colon H^*(X) \to H^*(A)$ is surjective. Further, show that if X deformation retracts to A then i^* is an isomorphism.*

4.13. **Exercise.** *Show that if we assume that the product in Theorem 4.5 (iii) is finite, then the statement of Theorem 4.5 (iii) follows from the existence of the Mayer-Vietoris sequence.*

4.14. It is sometimes convenient to consider a minor variant of cohomology: let X be a space and let $\varepsilon \colon X \to \mathrm{pt}$ be the obvious map. Define the *reduced cohomology* groups of X by:

$$\tilde{H}^q(X) = \operatorname{coker}(\varepsilon^* \colon H^q(\mathrm{pt}) \to H^q(X)).$$

As $H_q(\mathrm{pt})$ is trivial for all $q \neq 0$, we have

$$H^q(X) = \tilde{H}^q(X) \quad \text{for } q \neq 0 \quad \text{and} \quad H^0(X) \simeq \tilde{H}^0(X) \oplus \mathbf{Z}.$$

4.15. **Exercise.** *With the notation of Theorem 4.5, prove that if $A \cap B$ is not empty, then the Mayer-Vietoris sequence induces an exact sequence*

$$\cdots \xrightarrow{\delta} \tilde{H}^q(X) \xrightarrow{\left(\begin{smallmatrix} a^* \\ b^* \end{smallmatrix}\right)} \tilde{H}^q(A) \oplus \tilde{H}^q(B) \xrightarrow{(\, \alpha^* \;\, -\beta^* \,)} \tilde{H}^q(A \cap B) \xrightarrow{\delta} \tilde{H}^{q+1}(X) \to \cdots$$

4.16. **Exercise.** *Suppose X is the union of open sets $U_1, \ldots, U_n$ such that each intersection $U_{i_1} \cap \cdots \cap U_{i_k}$ is either empty or has trivial reduced cohomology groups. Show that $\tilde{H}^i(X) = 0$ for $i \geq n - 1$.*

4.17. **Example.** Let $D_1, D_2 \subset S^n$ be the complements of the north and south pole respectively. Then $D_1 \cap D_2$ deformation retracts to the equator. As D_1 and D_2 are contractible and $D_1 \cap D_2$ is homotopy equivalent to S^{n-1}, using the Mayer-Vietoris sequence we infer $\tilde{H}^q(S^n) \simeq \tilde{H}^{q-1}(S^{n-1})$. It follows

$$\tilde{H}^q(S^n) = \begin{cases} \mathbf{Z} & \text{if } q = n; \\ 0 & \text{otherwise.} \end{cases}$$

4.18. Let's record some consequences of these computations. We claim that $\mathbf{R}^n$ is not homeomorphic to $\mathbf{R}^m$ for $m \neq n$. Assume otherwise, then $\mathbf{R}^n - \{x\}$ is homeomorphic to $\mathbf{R}^m - \{y\}$ (for appropriate points x and y). But the former space is homotopy equivalent to S^{n-1} and the latter to S^{m-1}. The spaces S^{n-1} and S^{m-1} have different cohomology groups for $m \neq n$. This contradicts the homotopy invariance of cohomology.

4.19. The sphere $S^n = \partial D^{n+1}$ is not a retract of the disk D^{n+1}. As otherwise we would obtain a surjective map $H^*(D^{n+1}) \twoheadrightarrow H^*(S^n)$ which is clearly impossible. This further implies:

4.20. **Theorem** (Brouwer fixed point theorem)**.** *Every map $f\colon D^{n+1} \to D^{n+1}$ has a fixed point, i.e., there exists $x \in D^{n+1}$ such that $f(x) = x$.*

Proof. Assume otherwise. Then for each $x \in D^{n+1}$ there is a unique line ℓ_x passing through x and $f(x)$. This line ℓ_x meets $S^n = \partial D^{n+1}$ in exactly two points. Sliding x along ℓ_x towards $f(x)$ retracts D^{n+1} onto S^n. This is a contradiction. □

4.21. **Exercise.** *Explicitly give the retraction mentioned in the proof.*

4.22. **Exercise.** *The (unreduced) suspension of a space X is the space SX obtained by collapsing $X \times \{0\} \subset X \times I$ and $X \times \{1\} \subset X \times I$ to (distinct) points. Show that*

$$\tilde{H}^q(SX) \simeq \tilde{H}^{q-1}(X).$$

4.23. **Exercise.** *Show that*

$$H^q(S^1 \times S^1) = \begin{cases} \mathbf{Z} & \text{if } q = 0, 2; \\ \mathbf{Z} \oplus \mathbf{Z} & \text{if } q = 1; \\ 0 & \text{otherwise.} \end{cases}$$

4.24. **Exercise.** *Construct a canonical isomorphism*

$$H^q(X \times S^n) \simeq H^q(X) \oplus H^{q-n}(S^n)$$

for all q and n, where $H^q = 0$ for $q < 0$ by definition.

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