PRACTICE EXERCISES, LIMITS/COLIMITS, DEGREE

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Contents

1. Some practice with exact sequences

1.1. **Exercise.** *Suppose*

$$0 \to \mathbf{Z} \to A \to \mathbf{Z} \to 0$$

is an exact sequence of abelian groups. What can you say about A?

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is an exact sequence of abelian groups. What can you say about A?

1.8. **Exercise.** *Suppose*

$$0 \to A \to \mathbf{Z} \to B \to 0$$

is an exact sequence of abelian groups. What can you say about A and B?

1.9. **Exercise.** *Suppose*

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is an exact sequence of abelian groups. What can you say about A and B?

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is an exact sequence of abelian groups. What can you say about A *and* B*?*

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is an exact sequence of abelian groups. What can you say about A *and* B*?*

1.16. **Exercise.** *Compute the kernel and cokernel of the following maps of abelian groups*

$$\mathbf{Z} \oplus \mathbf{Z} \xrightarrow{\left(\begin{smallmatrix} \mathrm{id} & -\mathrm{id} \\ \mathrm{id} & -\mathrm{id} \end{smallmatrix}\right)} \mathbf{Z} \oplus \mathbf{Z}, \quad \mathbf{Z} \oplus \mathbf{Z} \xrightarrow{\left(\begin{smallmatrix} \mathrm{id} & \mathrm{id} \\ -\mathrm{id} & -\mathrm{id} \end{smallmatrix}\right)} \mathbf{Z} \oplus \mathbf{Z}.$$

The meaning of the 'matrix' notation above should be clear from the following examples. Suppose $f, g \colon \mathbf{Z} \to \mathbf{Z}$ *are group homomorphisms, then*

$$\mathbf{Z} \xrightarrow{\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)} \mathbf{Z} \oplus \mathbf{Z} \textit{ is the map } a \mapsto (f(a), g(a)),$$

$$\mathbf{Z} \oplus \mathbf{Z} \xrightarrow{\left(\begin{smallmatrix} f & g \end{smallmatrix}\right)} \mathbf{Z} \textit{ is the map } (a, b) \mapsto f(a) + g(b),$$

$$\mathbf{Z} \oplus \mathbf{Z} \xrightarrow{\left(\begin{smallmatrix} f & \mathrm{id} \\ -\mathrm{id} & g \end{smallmatrix}\right)} \mathbf{Z} \oplus \mathbf{Z} \textit{ is the map } (a, b) \mapsto (f(a) + b, -a + g(b)).$$

2. Limits and colimits

2.1. Let I be a small category (i.e., the objects of I form a set as opposed to just a class). Let $\mathcal{C}$ be any category. An I-shaped *diagram* in $\mathcal{C}$ is a functor $D \colon I \to \mathcal{C}$. A morphism $D \to D'$ of I shaped diagrams is a natural transformation, and we have the category $\mathcal{C}^I$ of I-shaped diagrams in $\mathcal{C}$. Every object X of $\mathcal{C}$ determines the *constant diagram* $\underline{X}$ that sends each object of I to X and sends each morphism of I to id_X. A *cone* of an I-shaped diagram D is an object X of $\mathcal{C}$ together with a morphism of diagrams $\underline{X} \to D$. The *limit*, denoted $\varprojlim D$ is a universal (final) cone of D. That is, if $f \colon \underline{Y} \to D$ is a cone of the diagram D, then there is a *unique* map $g \colon \underline{Y} \to \varprojlim D$ such that $f = i \circ g$, where $i \colon \varprojlim D \to D$ is the diagram morphism part of the data of the cone $\varprojlim D$.

2.2. The dual notion, obtained by reversing all the arrows in the definition of a limit, is that of *colimit* of a diagram D, denoted $\varinjlim D$. That is, one defines a *cocone* as an object X of $\mathcal{C}$ together with a morphism of diagrams $D \to \underline{X}$. Then the colimit is a universal (initial) cocone of D: if $f\colon D \to \underline{Y}$ is a cocone of the diagram, then there is a *unique* map $g\colon \varinjlim D \to \underline{Y}$ such that $f = g \circ i$, where $i\colon D \to \varinjlim D$ is the diagram morphism part of the data of the cocone $\varinjlim D$.

2.3. Limits and colimits, if they exist, are unique up to canonical isomorphism.

2.4. **Example.** Let I be the category

$$\bullet \qquad \bullet$$

where a '$\bullet$' denotes an object and the morphisms are the identity morphisms plus the arrows shown (composition given in the obvious way). Then limits indexed by I are called *products*. For instance, a product in the category of sets is the usual cartesian product. Colimits indexed by I are called *coproducts*. In the category of sets coproducts are given by disjoint unions.

2.5. **Example.** Let I be the empty category. A limit indexed by I is called an *initial* object. A colimit indexed by I is called a *final* object. In the category of sets the empty set is the initial object and the one point set is the final object. If a category has an initial and final object and both of these coincide then we call the object a *zero* object. Suppose $\mathcal{C}$ is a category with a zero object 0 (often such categories are called *pointed*). Then, by definition, there is a unique map $X \to 0$ for each $X \in \mathcal{C}$. Similarly, there is a unique map $0 \to X$ for each $X \in \mathcal{C}$. The composition $X \to 0 \to X$ is called the *zero map*. For instance, this notion corresponds to the usual one for the category of abelian groups.

2.6. **Exercise.** *Compute the colimits (if they exist) of the following diagrams in* **Top** *and in* **hTop***:*

$$\begin{array}{ccc} S^1 & \longrightarrow & D^2 \\ \downarrow & & \\ D^2 & & \end{array} \qquad \begin{array}{ccc} S^1 & \longrightarrow & \mathrm{pt} \\ \downarrow & & \\ D^2 & & \end{array} \qquad \begin{array}{ccc} S^1 & \longrightarrow & \mathrm{pt} \\ \downarrow & & \\ \mathrm{pt} & & \end{array}$$

where $S^1 \to D^2$ is the inclusion of S^1 as the boundary of D^2 and $S^1 \to \mathrm{pt}$ is the obvious map.

2.7. **Exercise.** *Let $i\colon A \hookrightarrow X$ be the inclusion of a subspace. Compute the colimit of the following diagram in* **Top**

$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ \downarrow & & \\ \mathrm{pt} & & \end{array}$$

What about the colimit of this same diagram in **hTop***?*

2.8. **Exercise.** *Let* **Vect** *be the category of vector spaces over some fixed field k. Compute the limits and colimits (if they exist) of the following diagrams in* **Vect***:*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \\ W & & \end{array} \qquad \begin{array}{ccc} & & U \\ & & \downarrow \\ V & \longrightarrow & W \end{array}$$

2.9. **Exercise.** *Compute the limits and colimits (if they exist) of the diagrams in the previous exercise, but this time assume that the diagrams are in* **Ab**.

2.10. **Exercise.** *Compute the limits and colimits (if they exist) of the diagrams in the previous exercise, but this time assume that the diagrams are in* **Top***.*

2.11. **Exercise.** *Compute the limits and colimits (if they exist) of the diagrams in the previous exercise, but this time assume that the diagrams are in* **hTop***.*

2.12. **Exercise.** *Formulate the notions of kernels, cokernels and images of a map (of abelian groups and/or vector spaces) in terms of limits and colimits.*

2.13. **Problem.** *Do* all *limits and colimits exist in the following categories:* **Top***,* **hTop***,* **Ab***?*

2.14. **Problem.** *There is an obvious functor* **Top** $\to$ **hTop***. Hence, we may view the* H^q*s as either functors* **Top** $\to$ **Ab** *or as functors* **hTop** $\to$ **Ab***. Do the functors* H^q: **Top** $\to$ **Ab** *send limits to colimits and send limits to colimits? What about if we ask the same question but with '***Top***' in the previous sentence replaced with '***hTop***'?*

3. Degree

3.1. Let $f\colon S^n \to S^n$ be a map. Then the endomorphism $f^*\colon \tilde{H}^n(S^n) \to \tilde{H}^n(S^n)$ may be identified with an integer, $\deg(f)$, called the *degree* of f. Certainly:

(i) $\deg(\mathrm{id}) = 1$;
(ii) $\deg(fg) = \deg(g) \cdot \deg(f)$;
(iii) if f is homotopic to g, then $\deg(f) = \deg(g)$;
(iv) if f is not surjective, then $\deg(f) = 0$;
(v) if f is a homotopy equivalence, then $\deg(f) = \pm 1$.

3.2. **Proposition.** *Define*

$$f_n\colon S^n \to S^n, \qquad (x_1, \dots, x_{n+1}) \mapsto (-x_1, x_2, \dots, x_{n+1}).$$

Then $\deg(f) = -1$.

Proof. Proceed by induction on n. The statement is easy to check for $n = 0$. Assume $n > 0$. Let D_1 and D_2 be the complement of the north pole and the south pole respectively. Then $f(D_i) \subseteq D_i$. Let $i\colon S^{n-1} \hookrightarrow D_1 \cap D_2$ be the inclusion of the equator. Then i is a homotopy equivalence and using the Mayer-Vietoris sequence we obtain a commutative diagram:

$$\begin{array}{ccccc}
\tilde{H}^{n-1}(S^{n-1}) & \xleftarrow[\sim]{i^*} & \tilde{H}^{n-1}(D_1 \cap D_2) & \xrightarrow[\sim]{\delta} & \tilde{H}^n(S^n) \\
\uparrow{\scriptstyle f^*_{n-1}} & & \uparrow{\scriptstyle f^*_n} & & \uparrow{\scriptstyle f^*_n} \\
\tilde{H}^{n-1}(S^{n-1}) & \xleftarrow[\sim]{i^*} & \tilde{H}^{n-1}(D_1 \cap D_2) & \xrightarrow[\sim]{\delta} & \tilde{H}^n(S^n)
\end{array}$$

Here all the horizontal arrows are isomorphisms. The result follows. □

3.3. **Exercise.** *Why does the result follow from the commutative diagram?*

3.4. **Exercise.** *Prove the* $n = 0$ *case.*

3.5. **Proposition.** *Define*

$$s\colon S^n \to S^n, \qquad (x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_{i-1}, -x_i, x_{i+1}, \dots, x_{n+1}).$$

Then $\deg(s) = -1$.

Proof. Let $h\colon S^n \to S^n$ be the map that interchanges the first and the i-th coordinate. As h is a homeomorphism, $\deg(h) = \pm 1$. Further, $s = h^{-1} f_n h$, where f_n is as in the previous Proposition. So $\deg(s) = \deg(h^{-1} f_n h) = \deg(f_n)$. □

3.6. Let X be a space. Recall that the *(unreduced) suspension*, SX, is the space obtained by collapsing $X \times \{0\} \subset X \times I$ and $X \times \{1\} \subset X \times I$ to (distinct) points. Let $f\colon X \to Y$ be a map. Define $Sf\colon SX \to SY$ by $(x, a) \mapsto (f(x), a)$. If f is homotopic to g, then Sf is homotopic to Sg. Hence, we obtain a functor $S\colon \mathbf{hTop} \to \mathbf{hTop}$.

3.7. **Proposition.** *Let $f\colon S^n \to S^n$ be a map. Then* $\deg(Sf) = \deg(f)$.

Proof. Exactly the same as that of Proposition 3.2. □

3.8. The *antipode map* $\alpha\colon S^n \to S^n$ is given by $x \mapsto -x$.

3.9. **Proposition.** $\deg(\alpha) = (-1)^{n+1}$.

Proof. The map α is the composition of $n+1$ maps of degree -1. So the result follows from Proposition 3.2. □

3.10. **Proposition.** *Let $f, g\colon S^n \to S^n$ be maps such that $f(x) \neq g(x)$ for all x. Then f is homotopic to αg, where $\alpha\colon S^n \to S^n$ is the antipode map.*

Proof sketch. As $f(x) \neq g(x)$, the line joining $f(x)$ and $-g(x)$ does not pass through the origin. Projecting this line out from the origin to the sphere gives the desired homotopy. □

3.11. **Exercise.** *Make this proof precise by explicitly giving the homotopy.*

3.12. **Corollary.** *Let $f\colon S^{2n} \to S^{2n}$ be a map. Then there is some $x \in S^{2n}$ such that $f(x) = \pm x$.*

Proof. Let $\alpha\colon S^{2n} \to S^{2n}$ be the antipode map. Suppose $f(x) \neq x$ for all x. Then f is homotopic to α. Similarly, if $f(x) \neq -x$ for all x, then f is homotopic to $\alpha^2 = \mathrm{id}$. But $\deg(\alpha) = -1 \neq \deg(\mathrm{id})$. In particular, α is not homotopic to the identity. The result follows. □

3.13. Let X and Y be spaces with chosen basepoints $x \in X$ and $y \in Y$. Then the *wedge* of X and Y, denoted $X \vee Y$, is the space obtained by identifying x and y in $X \sqcup Y$.

3.14. **Exercise.** *Express $X \vee Y$ as a limit or a colimit in* **Top**.

3.15. **Exercise.** *Show that under 'reasonable assumptions on the points x and y', there is a canonical isomorphism*

$$\tilde{H}^q(X \vee Y) \simeq \tilde{H}^q(X) \oplus \tilde{H}^q(Y).$$

Is the above statement true if we replaced reduced cohomology with unreduced cohomology?

3.16. Let $U_1, \ldots, U_k$ be disjoint open sets in S^n each homeomorphic to $\mathbf{R}^n$. Let $f\colon S^n \to Y$ be a map that maps $S^n - \bigcup_i U_i$ to a point $y \in Y$. Collapsing $S^n - \bigcup_j U_j$ to a point gives a space homeomorphic to the k-fold wedge of n-spheres. It follows that f factors as

$$S^n \xrightarrow{g} S^n \vee \cdots \vee S^n \xrightarrow{h} Y,$$

where the wedge sum has k terms. Let

$$i_j\colon S^n \hookrightarrow S^n \vee \cdots \vee S^n \quad \text{and} \quad p_j\colon S^n \vee \cdots \vee S^n \to S^n$$

be the inclusion of, and the projection on, the j-th factor respectively. Using the Mayer-Vietoris sequence one deduces

$$\begin{pmatrix} i_1^* \\ \vdots \\ i_k^* \end{pmatrix} : \tilde{H}^*(S^n \vee \cdots \vee S^n) \xrightarrow{\sim} \tilde{H}^*(S^n) \oplus \cdots \oplus \tilde{H}^*(S^n)$$

is an isomorphism. Its inverse is given by

$$\left(p_1^* \cdots p_k^* \right) : \tilde{H}^*(S^n) \oplus \cdots \oplus \tilde{H}^*(S^n) \xrightarrow{\sim} \tilde{H}^*(S^n \vee \cdots \vee S^n).$$

Let $g_j = p_j g$, $h_j = h i_j$ and $f_j = h_j g_j$. Then

$$f^* = g^* h^* = g^* \left(\sum_j p_j^* i_j^* \right) h^* = \sum_j f_j^*.$$

Hence, if $Y = S^n$, then $\deg(f) = \sum_j \deg(f_j)$. Note that f_j is f on U_j and maps the complement to y.

3.17. **Proposition.** *View S^1 as a subspace of $\mathbf{C}$. Let $k \in \mathbf{Z}$. Define $f \colon S^1 \to S^1$, $z \mapsto z^k$. Then $\deg(f) = k$.*

Proof sketch. It suffices to assume $k \geq 0$. The case $k = 0$ is obvious, so assume $k > 0$. We use the notation in the above discussion. Divide S^1 into k open arcs of equal length (these are the U_j above). Now f_j stretches the corresponding arc by a factor of k (in the same direction), wraps it around the circle and maps the complement to a point. Each f_j is homotopic to the identity. The result follows. □

3.18. **Exercise.** *Why does it suffice to assume $k \geq 0$?*

3.19. **Exercise.** *Make the above proof precise by explicitly defining the arcs and giving explicit homotopies between the f_j and the identity map.*

3.20. **Exercise.** *For $n \neq 0$, construct maps $S^n \to S^n$ of arbitrary degree.*

3.21. **Exercise.** *Either construct a surjective map $S^1 \to S^1$ of degree 0 or show that no such map can exist.*

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