SOME ABSTRACT NONSENSE. A BIT MORE ON DEGREE

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Contents

A lot of the 'abstract nonsense' below and in previous (and forthcoming) notes is more than what is strictly necessary to get an understanding of the basic notions of this course. In fact, much of it is a digression from the 'main topics'. Its inclusion in the notes partly reflects my own biases and is also partly prompted by some of the questions you asked in class. Try not to get bogged down in some of the technical details (this is not to say that the details aren't important).

1. Yoneda lemma

1.1. Let $F, G : \mathcal{A} \to \mathcal{B}$ be functors between categories $\mathcal{A}$ and $\mathcal{B}$. Recall that morphism of functors $\phi\colon F \to G$ consists of a morphism $\phi_X\colon F(X) \to G(X)$ for each $X \in \mathcal{A}$, such that $\phi_Y \circ F(f) = G(f) \circ \phi_X$ for every morphism $f\colon X \to Y$. Further, we use the terms 'functorial', 'natural' and 'canonical' as synonyms for 'a morphism of functors'. Denote the identity endomorphism of a functor F by $\mathbf{1}_F$.

1.2. A functor $F\colon \mathcal{A} \to \mathcal{B}$ is *full* if the map it induces on Hom sets is surjective; it is *faithful* if the induced map is injective. It is an *equivalence* if there exists a functor $G : \mathcal{B} \to \mathcal{A}$ such that FG and GF are canonically isomorphic to $\mathrm{id}_{\mathcal{B}}$ and $\mathrm{id}_{\mathcal{A}}$, respectively. In this situation the functors F and G are *mutually inverse equivalences.* An equivalence is necessarily full and faithful. Moreover:

1.3. **Proposition.** *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a full and faithful functor. Then F is an equivalence if and only if every object $Y \in \mathcal{B}$ is isomorphic to $F(X)$ for some $X \in \mathcal{A}$.*

Proof. Necessity is clear. Let's show sufficiency. For each $Y \in \mathcal{B}$ we pick a pair (X_Y, ϕ_Y) with $X_Y \in \mathcal{A}$ and $\phi_Y : Y \xrightarrow{\sim} F(X_Y)$ an isomorphism. Define $G\colon \mathcal{B} \to \mathcal{A}$ as follows. For an object $Y \in \mathcal{B}$, $G(Y) = X_Y$. If $f\colon Y \to Z$ is a morphism in $\mathcal{B}$, then $G(f)\colon X_Y \to X_Z$ is given by the formula

$$G(f) = F^{-1}(\phi_Z \circ f \circ \phi_Y^{-1}),$$

where by an abuse of notation F^{-1} is the isomorphism (of sets)

$$\mathrm{Hom}_{\mathcal{B}}(F(X_Y), F(X_Z)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(X_Y, X_Z)$$

given by the functor F. One checks easily that F and G are mutually inverse equivalences. $\square$

1.4. **Exercise.** *Show that the functor G in the proof is indeed a functor.*

1.5. **Exercise.** *Show that the functors F and G in the proof are mutually inverse equivalences.*

1.6. Let $\mathcal{A}$ be a category. Let **Set** be the category of sets. Let $\mathrm{Funct}(\mathcal{A}^{\mathrm{op}}, \mathbf{Set})$ be the category of contravariant functors $\mathcal{A} \to \mathbf{Set}$. A functor $F \in \mathrm{Funct}(\mathcal{A}^{\mathrm{op}}, \mathbf{Set})$ is *representable* if $F \simeq \mathrm{Hom}_{\mathcal{A}}(-, X)$ for some object $X \in \mathcal{A}$. In this situation, the object X is said to *represent* F. For $X \in \mathcal{A}$ we set $h_X = \mathrm{Hom}_{\mathcal{A}}(-, X)$, so that $h_X \in \mathrm{Funct}(\mathcal{A}^{\mathrm{op}}, \mathbf{Set})$. To ease notation, for $F, G \in \mathrm{Funct}(\mathcal{A}^{\mathrm{op}}, \mathbf{Set})$, write $\mathscr{H}om(F, G)$ for the set of natural transformations $F \to G$.

1.7. **Lemma** (Yoneda lemma)**.** *Let $X \in \mathcal{A}$ and let $F \in \mathrm{Funct}(\mathcal{A}^{\mathrm{op}}, \mathbf{Set})$. Then the map*

$$\mathscr{H}om(h_X, F) \to F(X),$$
$$\phi \mapsto \phi_X(\mathrm{id}_X)$$

is an isomorphism.

Proof. The inverse $F(X) \to \mathscr{H}om(h_X, F)$ is defined as follows. Let $u \in F(X)$. Then for $Y \in \mathcal{A}$, define the map $u_Y \colon \mathrm{Hom}_{\mathcal{A}}(Y, X) \to F(Y)$ by $f \mapsto (Ff)(u)$. The family u_Y, $Y \in \mathcal{A}$ defines a natural transformation $h_X \to F$. An easy computation shows that this gives the required inverse. □

1.8. **Exercise.** *Do the afore-mentioned 'easy computation'.*

1.9. **Problem.** *Meditate on the Yoneda lemma.*

1.10. **Corollary.** *The functor $\mathcal{A} \to \mathrm{Funct}(\mathcal{A}^{\mathrm{op}}, \mathbf{Set})$, $X \mapsto h_X$, is full and faithful.*

Proof. Apply the Yoneda lemma to $\mathscr{H}om(h_Y, h_X)$, $X, Y \in \mathcal{A}$. □

1.11. *Remark.* The functor $\mathcal{A} \to \mathrm{Funct}(\mathcal{A}^{\mathrm{op}}, \mathbf{Set})$, $X \mapsto h_X$, is called the Yoneda embedding. Morally, all that the abstract nonsense above says is that an object $X \in \mathcal{A}$ is completely determined by the morphisms into it. So, if you want to study an object, then study the morphisms into it.

2. Additive categories

2.1. A category $\mathcal{A}$ is *additive* if all Hom sets are equipped with an abelian group structure such that composition of morphisms is bilinear and if all finite products exist in $\mathcal{A}$. The empty product gives a terminal object in $\mathcal{A}$. For $X, Y \in \mathcal{A}$, the maps $X \xleftarrow{\mathrm{id}} X \xrightarrow{0} Y$ give a unique map $X \to X \times Y$. Similarly, there is a unique map $Y \to X \times Y$. Consequently, finite products coincide with the corresponding coproducts. In particular, the terminal object is also initial and is hence a zero object.

2.2. **Exercise.** *Verify the above assertion regarding the coincidence of finite products and coproducts in additive categories.*

2.3. Let $\mathcal{B}$ be another additive category. An *additive functor* $\mathcal{A} \to \mathcal{B}$ is a functor F such that $F(f + g) = F(f) + F(g)$ for all morphisms $f, g \in \mathcal{A}$.

3. Abelian categories

3.1. An additive category is *abelian* if it possesses all kernels, cokernels and if every monomorphism is the kernel of some morphism and every epimorphism is the cokernel of some morphism. Note: to make sense of this previous sentence you need to recall/formulate the notion of epimorphisms, monomorphisms, kernels and cokernels in an additive category (see the previous week's notes).

3.2. **Exercise.** *Give an example of an additive category that is not abelian.*

3.3. Let $\mathcal{A}$ be an abelian category. A sequence of maps $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_n} X_{n+1}$, in $\mathcal{A}$, is an *exact sequence* if the image of f_i is equal to the kernel of f_{i+1} for each $0 \le i < n$. An exact sequence $0 \to X \to Y \to Z \to 0$ is also referred to as a *short exact sequence.*

3.4. Let $\mathcal{B}$ be another abelian category. A functor $F\colon \mathcal{A} \to \mathcal{B}$ is *left exact* if for each exact sequence $0 \to X \to Y$ in $\mathcal{A}$, the sequence $0 \to F(X) \to F(Y)$ is exact in $\mathcal{B}$. Similarly, F is *right exact* if for each exact sequence $X \to Y \to 0$ in $\mathcal{A}$, the sequence $F(X) \to F(Y) \to 0$ is exact in $\mathcal{B}$. The functor F is *exact* if it is both left and right exact.

3.5. The *Grothendieck group* $K_0(\mathcal{A})$ of an abelian category $\mathcal{A}$ is the free abelian group on symbols $[X]$, $X \in \mathcal{A}$, modulo the relation $[X] = [X_1] + [X_2]$ for each short exact sequence $0 \to X_1 \to X \to X_2 \to 0$. Consequently, if $X^\bullet = \cdots \to X^i \to \cdots$ is a bounded complex in $\mathcal{A}$, then $\sum_i (-1)^i [X^i] = \sum_i (-1)^i [H^i(X^\bullet)]$ in $K_0(\mathcal{A})$. If $F\colon \mathcal{A} \to \mathcal{B}$ is an exact functor between abelian categories, then the map $[X] \mapsto [F(X)]$ is a group homomorphism $K_0(\mathcal{A}) \to K_0(\mathcal{B})$.

3.6. Let $\{L_i\}$ be a set of objects in $\mathcal{A}$ such that the classes $[L_i]$ comprise a basis of $K_0(\mathcal{A})$. Then for $M \in \mathcal{A}$, we write $[M : L_i]$ for the coefficient of L_i when $[M]$ is expanded in terms of the basis $\{[L_i]\}$, i.e., $[M] = \sum_i [M : L_i][L_i]$.

3.7. A *simple object* or an *object of length one* is an object $L \in \mathcal{A}$ such that any monomorphism $A \to L$ is either 0 or an isomorphism. For $n \ge 2$, *objects of length* n are inductively defined to be those objects X that fit into an exact sequence $0 \to X' \to X \to L \to 0$, with X' of length $n-1$ and L simple. If every object in $\mathcal{A}$ has finite length, then the Jordan-Hölder theorem holds in $\mathcal{A}$, i.e., for an object $X \in \mathcal{A}$, the length of X is well defined and the simple objects that occur in a 'composition series' of X are unique up to isomorphism and permutation.

3.8. **Exercise.** *Let* **Vect** *be the category of finite dimensional vector spaces over some fixed field k. What are the simple objects in* **Vect***? What is* $K_0(\mathbf{Vect})$*?*

3.9. **Exercise.** *Same questions as the previous exercise but replace* Vect *with the category of all vector spaces over k.*

4. Adjoint functors

4.1. Let $f_*\colon \mathcal{A} \to \mathcal{B}$ and $f^*\colon \mathcal{B} \to \mathcal{A}$ be functors. An *adjunction* (f^*, f_*) between f^* and f_* is the data of two natural transformations $\varepsilon\colon f^*f_* \to \mathrm{id}_{\mathcal{A}}$ and $\eta : \mathrm{id}_{\mathcal{B}} \to f_*f^*$ such that the compositions

$$f_* \xrightarrow{\eta \mathbf{1}_{f_*}} f_*f^*f_* \xrightarrow{\mathbf{1}_{f_*}\varepsilon} f_* \qquad \text{and} \qquad f^* \xrightarrow{\mathbf{1}_{f^*}\eta} f^*f_*f^* \xrightarrow{\varepsilon \mathbf{1}_{f^*}} f^* \tag{4.1.1}$$

are equal to the identity on f_* and f^*, respectively. The morphisms η and ε are the *unit* and *counit* of the adjunction respectively.

4.2. An adjunction gives an isomorphism, functorial in $A \in \mathcal{A}$ and $B \in \mathcal{B}$:

$$\alpha_{A,B}\colon \mathrm{Hom}_{\mathcal{A}}(f^*B, A) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{B}}(B, f_*A), \qquad \phi \mapsto \mathbf{1}_{f_*}\phi \circ \eta_B.$$

The inverse is given by $\psi \mapsto \varepsilon_A \circ \mathbf{1}_{f^*}\psi$. Conversely, such a functorial isomorphism $\alpha_{A,B}$ provides an adjunction (f^*, f_*). Namely, set $\varepsilon_A = \alpha^{-1}_{A,f_*A}(\mathrm{id}_{f_*A})$ and $\eta_B = \alpha_{f^*B,B}(\mathrm{id}_{f^*B})$.

4.3. If (f^*, f_*) is an adjunction, then the functor f^* is said to be *left adjoint* to f_* and the functor f_* is said to be *right adjoint* to f^*.

4.4. **Exercise.** *Let* $\mathrm{For}\colon \mathbf{Ab} \to \mathbf{Set}$ *be the functor that associates to a group its underlying set. Let* $\mathrm{Free}\colon \mathbf{Set} \to \mathbf{Ab}$ *be the functor that associates to a set the free abelian group generated by it. Show that* Free *is left adjoint to* For.

4.5. **Exercise.** *Let* $F\colon \mathcal{A} \to \mathcal{B}$ *be an additive functor between abelian categories. Suppose a left adjoint for* F *exists. Then show that* F *is left exact.*

4.6. **Exercise.** *Same question as the previous exercise but replace 'left' everywhere by 'right'.*

5. Back to topology: Borsuk-Ulam

5.1. **Lemma.** *Let* $f\colon S^1 \to S^1$ *be a map such that* $f(-x) = -f(x)$ *for all* $x \in S^1$. *Then the degree of* f *is odd.*

5.2. **Exercise.** *Prove this!* Hint: *To get some intuition, it might help to remember that morally 'degree = winding number = number of times the map runs around the circle'.*

5.3. **Exercise.** *Let* $f\colon S^1 \to S^1$ *be a map such that* $f(-x) = f(x)$ *for all* $x \in S^1$. *Prove that the degree of* f *is even.*

5.4. **Problem.** *Let* $f\colon S^n \to S^n$ *be a map such that* $f(-x) = -f(x)$ *for all* $x \in S^n$. *Prove that the degree of* f *is odd.*

5.5. **Problem.** *Let* $f\colon S^n \to S^n$ *be a map such that* $f(-x) = f(x)$ *for al* $x \in S^n$. *Prove that the degree of* f *is even.*

5.6. **Lemma.** *There is no continuous map* $f\colon S^2 \to S^1$ *such that* $f(-x) = -f(x)$ *for all* $x \in S^2$.

Proof. By way of contradiction assume that $f\colon S^2 \to S^1$ is a map such that $f(-x) = -f(x)$ for all $x \in S^2$. Define

$$\begin{aligned} g\colon D^2 &\to S^1, \\ (x, y) &\mapsto f(x, y, \sqrt{1 - x^2 - y^2}). \end{aligned}$$

Restricting g to $\partial D^2 = S^1$ we obtain a map $h\colon S^1 \to S^1$ that satisfies $h(-z) = -h(z)$ for all $z \in S^1$. The previous lemma implies that the degree of h is odd. But this is absurd, since h factors thorugh g and hence must have degree 0. □

5.7. **Proposition** (Borsuk-Ulam)**.** *Let* $f\colon S^2 \to \mathbf{R}^2$ *be a map. Then there exists* $x \in S^2$ *such that* $f(-x) = f(x)$.

Proof. Assume otherwise. Then the map

$$\begin{aligned} g\colon S^2 &\to S^1, \\ x &\mapsto \frac{f(x) - f(-x)}{|f(x) - f(-x)|}, \end{aligned}$$

satisfies $g(-x) = -g(x)$. This contradicts the previous lemma. □

5.8. *Remark.* So there are always two antipodal points on the earth with the same temperature and humidity (assuming that the temperature and humidity are continuous functions of location).

5.9. **Exercise.** *Formulate and prove the higher dimensional analogue of Borsuk-Ulam.*

5.10. **Exercise.** *Does Borsuk-Ulam hold for the torus? That is, if* $f\colon S^1 \times S^1 \to \mathbf{R}^2$ *is a map, then must there exist* $(x, y) \in S^1 \times S^1$ *such that* $f(-x, -y) = f(x, y)$*?*

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