SOME OPERATIONS ON SHEAVES

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Contents

All sheaves are assumed to be abelian sheaves. For a space X we write $\mathrm{Sh}(X)$ for the category of sheaves on X.

At first reading I recommend that you skip straight to §9 and §10. You may also want to glance at the section on adjoint functors in the notes for week 3. Further, when you do start reading the sections in this set of notes sequentially, keep a copy of §12 on the side and try to do these additional exercises as you go along.

1. Pushforward

1.1. Let $f\colon X \to Y$ be a map between spaces and let F be a sheaf on X. Define a presheaf f_*F on Y by

$$f_*F(V) = F(f^{-1}(V))$$

for an open subset V of Y. If U is an open subset of V, define the restriction map $f_*F(V) \to f_*F(U)$ by $s \mapsto \mathrm{res}^{f^{-1}(V)}_{f^{-1}(U)}(s)$. The presheaf f_*F is a sheaf.

1.2. If $\alpha\colon F \to G$ is a morphism of sheaves on X, define a morphism $f_*(\alpha)\colon f_*F \to f_*G$ as follows: a section of f_*F over an open subset V of X is, by definition, a section $s \in F(f^{-1}(V))$ and $f_*(\alpha)$ is given by mapping s to $\alpha_{f^{-1}(V)}(s)$ which is a section of $G(f^{-1}(V)$. The latter is, by definition a section of $f_*G(V)$. Consequently, we obtain a functor

$$f_*\colon \mathrm{Sh}(X) \to \mathrm{Sh}(Y).$$

The functor f_* is called the *pushforward* or the *direct image* along the map f.

1.3. It is clear that if $g\colon Y \to Z$ is another map between spaces, then $g_* f_* = (gf)_*$.

1.4. **Example.** Let X be a space and let $a\colon X \to \mathrm{pt}$ be the obvious map. If F is a sheaf on X, then $a_* F = F(X) = \Gamma(X; F)$. In this special case a_* is also called the *global sections* functor. The reason for this terminology should be obvious.

1.5. **Example.** Let X be a space and let $x \in X$. Then the pushforward of the constant sheaf, $\underline{\mathbf{Z}}_x$ on $\{x\}$, along the inclusion map $i\colon \{x\} \hookrightarrow X$, is given by

$$i_* \underline{\mathbf{Z}}_x(U) = \begin{cases} \mathbf{Z} & \text{if } x \in U; \\ 0 & \text{otherwise,} \end{cases}$$

where U is an open subset of X. The sheaf $i_* \underline{\mathbf{Z}}_x$ is called the *skyscraper sheaf* supported on x.

1.6. **Example.** More generally, if $f\colon X \hookrightarrow Y$ is the inclusion of a subspace and F is a sheaf on X, then

$$f_* F(U) = F(U \cap X),$$

for open subsets U of Y.

1.7. **Proposition.** *Let $f\colon X \to Y$ be a map between spaces. Let*

$$0 \to F \to G \to H$$

be an exact sequence of sheaves on X. Then the sequence

$$0 \to f_* F \to f_* G \to f_* H$$

is exact.

Proof. As $0 \to F \to G \to H$ is exact, the sequence $0 \to F(U) \to G(U) \to H(U)$ is exact for every open subset U of X. The result follows. □

1.8. **Exercise.** *Show, via an example, that if $0 \to F \to G \to H \to 0$ is an exact sequence of sheaves on a space X, then it is not necessarily true that $0 \to f_* F \to f_* G \to f_* H \to 0$ is exact.*

1.9. **Proposition.** *Let*

$$i\colon Z \hookrightarrow X$$

be the inclusion of a closed subspace. Then the functor i_* is exact. That is, if

$$0 \to F \to G \to H \to 0$$

is an exact sequence of sheaves on Z, then

$$0 \to i_* F \to i_* G \to i_* H \to 0$$

is exact.

Proof. Courtesy of the previous result, we only need to show that $i_* G \to i_* H \to 0$ is exact. Let $z \in Z$ and let s be a section of $i_* H$ over an open neighborhood U of z. Then s is a section of H over $U \cap Z$. As $G \to H \to 0$ is exact, there is some $t_z \in G_z$ that maps to s_z. It follows that $(i_* G)_z \to (i_* H)_z \to 0$ is exact. If $x \in X - Z$, then the stalks $(i_* G)_x$ and $(i_* H)_x$ vanish. Hence, $i_* G \to i_* H \to 0$ is exact at all stalks. □

1.10. **Exercise.** *Is the analogue of the above result true for inclusions of open subspaces?*

1.11. The above discussion on exactness is summarized by saying that the functor f_* is *left exact* but not generally exact. However, if $f = i$ is the inclusion of a closed subspace, then i_* is exact.

2. Pullback

2.1. Let $f\colon X \to Y$ be a map between spaces and let F be a sheaf on Y. Let U be an open subset of X. Define a presheaf $f^t F$ on X via the assignment

$$\Gamma(U; f^t F) = \varinjlim_{V \supseteq f(U)} F(V),$$

where V runs through the open neighborhoods of $f(U)$ and the maps in the limit diagram are the restriction maps (I am going to stop repeating this about the maps, it should be clear by now that the maps in limit diagrams of this form are going to be the restriction maps. Constantly repeating this is starting to make for some very tedious exposition). The restriction maps for this presheaf are defined in the obvious way. Note: if $f(U)$ is open in Y, then $\varinjlim_{V \supseteq f(U)} F(V) = F(f(U))$. The presheaf $f^t F$ is usually *not* a sheaf. The sheafification of this presheaf, denoted $f^{-1}F$, defines a functor

$$f^{-1}\colon \mathrm{Sh(Y)} \to \mathrm{Sh(X)},$$

called the *pullback* or the *inverse image* along the map f. The functor f^{-1} is defined on morphisms in the evident way.

2.2. **Exercise.** *Find an example demonstrating that the presheaf $f^t F$ is not necessarily a sheaf.*

2.3. **Example.** Let $j\colon U \hookrightarrow X$ be the inclusion of an open subset and let F be a sheaf on X. Then the assignment

$$V \mapsto \varinjlim_{V' \supseteq V} F(V') = F(V),$$

V open in U (and hence open in X), is already a sheaf. Thus,

$$j^{-1}F(V) = F(V).$$

2.4. The definition of f^{-1} implies that for $x \in X$,

$$(f^{-1}F)_x = F_{f(x)}. \tag{2.4.1}$$

2.5. **Exercise.** *Verify this assertion.*

2.6. Let $g\colon Y \to Z$ be another map of spaces. Then one checks, either directly from the definitions or using the results of the next section, that $(gf)^* = f^* g^*$.

2.7. **Exercise.** *Verify this.*

2.8. *Remark.* Strictly speaking, the equality above is really a canonical isomorphism. However, this canonical isomorphism is 'coherent': there is no ambiguity in pulling back along three composable maps, all the diagrams obtained from these canonical isomorphisms commute, etc. So nothing is lost by replacing the isomorphism with equality.

2.9. **Example.** Let F be a sheaf on a space X, let $x \in X$ and let $i\colon \{x\} \hookrightarrow X$ be the inclusion map. Then $i^{-1}F = F_x$.

2.10. **Example.** Let F be a sheaf on a space X and let $f\colon W \hookrightarrow X$ be the inclusion of a subspace. Then the sheaf $f^{-1}F$ is called the *restriction* of F to W. This sheaf is often denoted $F|_W$.

2.11. **Example.** Let $a\colon X \to \mathrm{pt}$ be the obvious map. Then $a^{-1}\underline{\mathbf{Z}}_{\mathrm{pt}}$ is canonically isomorphic to $\underline{\mathbf{Z}}_X$.

2.12. **Proposition.** *Let $f\colon X \to Y$ be a map between spaces and let*

$$0 \to F \to G \to H \to 0$$

be an exact sequence of sequence of sheaves on Y. Then the sequence

$$0 \to f^{-1}F \to f^{-1}G \to f^{-1}H \to 0$$

is exact.

Proof. It suffices to check exactness at stalks. This follows from (2.4.1). □

2.13. In other words, f^{-1} is an exact functor.

2.14. **Warning.** In the literature, the functor f^{-1} is sometimes denoted by f^*. Some care needs to be exercised to avoid confusion, since the functor f^* is a different beast from 'pullback' maps on cohomology groups. To further add to the confusion, if one is dealing with sheaves in algebraic geometry (à la coherent/quasi-coherent sheaves) then f^* is what we are calling f^{-1} composed with tensoring with the structure sheaf (whatever that may be). On the same note, f_* should not be confused with 'pushforward' maps on homology groups.

3. The adjunction (f^{-1}, f_*)

3.1. Let $f\colon X \to Y$ be a map between spaces. Let F be a sheaf on Y and let U be an open subset of Y. Then

$$\Gamma(U; f_*f^{-1}F) = \Gamma(f^{-1}(U); f^{-1}F).$$

Clearly $U \supseteq f(f^{-1}(U))$. So, using the defining property of a colimit we obtain a map

$$F(U) \to \varinjlim_{V \supseteq f(f^{-1}(U))} F(V) = \Gamma(f^{-1}(U), f^t F).$$

Composing this with the canonical morphism from a presheaf to its sheafification, we obtain a morphism of sheaves

$$\eta_F\colon F \to f_*f^{-1}F.$$

3.2. Note that a section $s \in F(U)$ maps to 0 under η_F if and only if there is an open subset V of U containing $f(f^{-1}(U))$ such that $s|_V = 0$.

3.3. The family of morphisms η_F, $F \in \mathrm{Sh}(Y)$, defines a natural transformation $\eta\colon \mathrm{id} \to f_*f^{-1}$.

3.4. **Example.** Let $j\colon U \hookrightarrow X$ be the inclusion of an open subset. Then we have the following description of the unit map $\eta\colon \mathrm{id} \to j_*j^{-1}$. Let F be a sheaf on X and let V be an open subset of X. Then

$$j_*j^{-1}F(V) = F(U \cap V)$$

and if $s \in F(V)$ then η_F maps s to $s|_{U\cap V}$.

3.5. Let G be a sheaf on X and let V be an open subset of X. Then

$$f^t f_* G(V) = \varinjlim_{V' \supseteq f(V)} f_*G(V') = \varinjlim_{V' \supseteq f(V)} G(f^{-1}(V')),$$

where V' runs through open subsets of Y containing $f(V)$. Certainly, if V' contains $f(V)$, then $f^{-1}(V')$ contains V. Hence, using the universal property of the colimit we obtain a canonical map

$$f^t f_* G(V) \to G(V).$$

This gives a morphism from the presheaf $f^t f_* F$ to G. Using the universal property of sheafification we obtain a morphism of sheaves

$$\varepsilon_G\colon f^{-1}f_*G \to G.$$

3.6. The morphism ε_G is easy to describe at the level of stalks. Let $x \in X$. Then

$$(f^{-1}f_*G)_x = (f_*G)_{f(x)} = \varinjlim_{V \ni f(x)} G(f^{-1}(V)).$$

The map $(f^{-1}f_*G)_x \to G_x$ induced by ε_G is the evident one given by the universal property of the colimit.

3.7. In particular, if $f\colon X \hookrightarrow Y$ is the inclusion of a subspace, and F is a sheaf on X, then $\varepsilon_F\colon f^{-1}f_*F \to F$ is an isomorphism.

3.8. The family of morphisms ε_G, $G \in \mathrm{Sh}(X)$, defines a natural transformation $\varepsilon\colon f^{-1}f_* \to \mathrm{id}$. With a bit of patience one checks that the compositions

$$f_*G \xrightarrow{\eta_{f_*G}} f_*f^{-1}f_*G \xrightarrow{f_*(\varepsilon_G)} f_*G \quad \text{and} \quad f^{-1}F \xrightarrow{f^{-1}(\eta_F)} f^{-1}f_*f^{-1}F \xrightarrow{\varepsilon_{f^{-1}F}} f^{-1}F$$

are equal to the identity on f_*G and $f^{-1}F$ respectively. In other words, f^{-1} is left adjoint to f_*.

3.9. **Exercise.** *Check this!*

4. Support of a sheaf

4.1. Let F be a sheaf on a space X. Let $U \subseteq X$ be an open subset. Recall that the support of a section $s \in F(U)$ is the complement in U of the union of open sets on which s restricts to 0. The *support* of F, denoted $\operatorname{supp} F$, is defined to be the complement in X of the union of all open subsets U of X such that $F|_U = 0$.

4.2. It is clear that if $x \notin \operatorname{supp} F$ then $F_x = 0$. However, the converse is not true.

4.3. **Exercise.** *Find an example demonstrating that the reverse implication is false.*

4.4. **Proposition.** *We have*

$$\operatorname{supp} F = \overline{\{x \in X \mid F_x \neq 0\}}.$$

Proof. Let $x \in \operatorname{supp} F$. Then for every open neighborhood U of x the restriction $F|_U$ is non-zero. In particular, every open neighborhood of x contains a point at which the stalk of F is non-zero. Hence, $x \in \overline{\{x \in X \mid F_x \neq 0\}}$. Conversely, let $y \in \overline{\{x \in X \mid F_x \neq 0\}}$. Then every open neighborhood of y contains a point at which the stalk of F does not vanish. Hence, $F|_U \neq 0$ for every open neigborhood U of x. Thus, $y \in \operatorname{supp} F$. □

4.5. **Exercise.** *Let $j\colon U \hookrightarrow X$ be the inclusion of an open subset. Let F be a sheaf on U. Is it true that $\operatorname{supp} j_*F$ is contained in U? What about if we replace the open inclusion by a closed inclusion?*

4.6. **Exercise.** *Let Y be a space and let $X \subseteq Y$ be a subspace. Let $\mathrm{Sh}_X(Y)$ be the category of sheaves on Y with support contained in X. Prove that the categories $\mathrm{Sh}(X)$ and $\mathrm{Sh}_X(Y)$ are equivalent.*

5. Extension by zero

5.1. Let $i\colon Y \hookrightarrow X$ be the inclusion of a closed subspace and F a sheaf on Y. Then the sheaf i_*F on X is the unique (up to isomorphism) sheaf F_Y on X such that

$$F_Y|_Y = F \quad \text{and} \quad F_Y|_{X-Y} = 0.$$

In particular, $(F_Y)_x = 0$ if $x \notin Y$, and $(F_Y)_y = F_y$ if $y \in Y$.

5.2. The above statements are *not* true if we replace 'closed' by 'open'. We will now rectify this situation.

5.3. Let $j\colon U \hookrightarrow X$ be the inclusion of an open subset and let F be a sheaf on U. Define a sheaf $j_!F$ on X by

$$j_!F(V) = \begin{cases} F(V) & \text{if } V \subseteq U; \\ 0 & \text{otherwise,} \end{cases}$$

where V is an open subset of X. Restriction maps are defined in the obvious way. Then we have a functor

$$j_!\colon \mathrm{Sh}(U) \to \mathrm{Sh}(X),$$

with $j_!$ defined on morphisms in the evident way. The sheaf $j_!F$ is called the *extension by zero* of F to X. It is clear that if $j'\colon U' \hookrightarrow U$ is the inclusion of an open subset, then $(j'j)_! = j'_!j_!$.

5.4. The sheaf $j_!F$ is the unique (up to isomorphism) sheaf on F_U on X such that

$$F_U|_U = F \quad \text{and} \quad F_U|_{X-U} = 0.$$

The sheaf $F_U = j_!F$ is called the *extension by zero* of F to X.

5.5. **Proposition.** *The functor $j_!$ is exact. That is, if*

$$0 \to F \to G \to H \to 0$$

is an exact sequence of sheaves on U, then

$$0 \to j_!F \to j_!G \to j_!H \to 0$$

is an exact sequence of sheaves on X.

Proof. Exactness at the level of stalks is clear. □

6. The adjunction $(j_!, j^*)$

6.1. Let X be a space, let $j\colon U \hookrightarrow X$ be the inclusion of an open subspace and let F be a sheaf on U, and let $V \subseteq U$ be an open subset. Then

$$j^t j_!F(V) = F(V).$$

As j is an open inclusion, the presheaf $j^t j_!F$ is a sheaf. Hence, we obtain a canonical isomorphism

$$\eta_F\colon F \xrightarrow{\sim} j^*j_!.$$

6.2. Now let G be a sheaf on X, and let V' be an open subset of X. Then

$$j_!j^tG(V') = \begin{cases} G(V') & \text{if } V' \subseteq U; \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, we obtain a map $j_!j^tF \to F$. Now j^tF is canonically isomorphic to j^*F. So we obtain a canonical morphism

$$\varepsilon_F\colon j_!j^*F \to F.$$

The families η_F, $F \in \mathrm{Sh}(U)$, and ε_G, $G \in \mathrm{Sh}(X)$, define natural transformations

$$\eta\colon \mathrm{id} \to j^*j_! \qquad \text{and} \qquad \varepsilon\colon j_!j^* \to \mathrm{id}.$$

6.3. The compositions

$$j^*G \xrightarrow{\eta_{j^*G}} j^*j_!j^*G \xrightarrow{j^*(\varepsilon_G)} j^*G \quad \text{and} \quad j_!F \xrightarrow{j_!(\eta_F)} j_!j^*j_!F \xrightarrow{\varepsilon_{j_!F}} j_!F$$

are equal to the identity on j^*G and $j_!F$ respectively. In other words, $j_!$ is left adjoint to j^*.

6.4. **Exercise.** *Verify this assertion.*

7. Sections with support in a closed subspace

7.1. Let $i\colon Z \hookrightarrow X$ be the inclusion of a closed subspace and let F be a sheaf on X. Define a sheaf F' by

$$F'(U) = \{s \in F(U) \,|\, \operatorname{supp} s \subseteq Z\}. \tag{7.1.1}$$

As usual, restriction maps are defined in the obvious way.

7.2. **Exercise.** *Show that F' is indeed a sheaf.*

7.3. Define

$$i^!\colon \mathrm{Sh}(X) \to \mathrm{Sh}(Z), \qquad F \mapsto i^*F'.$$

7.4. **Proposition.** *The functor $i^!$ is left exact. That is, if*

$$0 \to F \to G \to H$$

is an exact sequence of sheaves on X, then

$$0 \to i^!F \to i^!G \to i^!H$$

is exact.

7.5. **Exercise.** *Verify this.*

8. The adjunction $(i_*, i^!)$

8.1. Let $i\colon Z \hookrightarrow X$ be the inclusion of a closed subspace. Let F be a sheaf on X and let F' be as in (7.1.1). Let $U \subseteq X$ be an open subset. Then

$$\Gamma(U; i_*i^!F) = \Gamma(U \cap Z; i^!F) = \Gamma(U \cap Z; i^*F').$$

Now

$$\Gamma(U \cap Z; i^tF') = \varinjlim_{V \supseteq U \cap Z} F'(V) = F'(U \cap Z).$$

Hence, we obtain a canonical isomorphism

$$F' \xrightarrow{\sim} i_*i^!F.$$

Furthermore, there is an obvious (injective) map $F' \to F$. Consequently, we obtain a canonical morphism

$$\varepsilon_F\colon i_*i^!F \to F.$$

8.2. Now let G be a sheaf on Z and let $(i_*G)'$ be as in (7.1.1). Then $(i_*G)' = i_*G$. Consequently, if V is an open subset of Z, then

$$\Gamma(V; i^t(i_*G)') = \varinjlim_{U \supseteq V} \Gamma(U; i_*G) = \varinjlim_{U \supseteq V} G(U \cap Z) = G(V).$$

Hence, we obtain a map $G \to i^t(i_*G)'$ which induces a canonical isomorphism

$$\eta_G\colon G \to i^!i_*G.$$

The families η_G, $G \in \mathrm{Sh}(Z)$, and ε_F, $F \in \mathrm{Sh}(X)$, define natural transformations

$$\eta\colon \mathrm{id} \to i^!i_* \quad \text{and} \quad \varepsilon\colon i_*i^! \to \mathrm{id}.$$

8.3. The compositions

$$i^!F \xrightarrow{\eta_{i^!F}} i^!i_*i^!F \xrightarrow{i^!(\varepsilon_F)} i^!F \quad \text{and} \quad i_*G \xrightarrow{i_*(\eta_G)} i_*i^!i_*G \xrightarrow{\varepsilon_{i_*G}} i_*G$$

are equal to the identity on $i^!F$ and i_*G respectively. In other words, i_* is left adjoint to $i^!$.

8.4. **Exercise.** *Verify this assertion.*

9. Recollement

9.1. **Exercise.** *Meditate on why the title of this section is what it is.*

9.2. The myriad relations between pushing forward and pulling back can be a bit overwhelming at first. Here's a quick summary to help you keep track of some of the important ones.

9.3. For every map $f\colon X \to Y$ between spaces there are functors

$f_*\colon \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$	pushforward	(left exact);
$f^{-1}\colon \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$	pullback	(exact).

9.4. Let $j\colon U \hookrightarrow X$ be the inclusion of an open subspace and let $i\colon Z \hookrightarrow X$ be the inclusion of the closed complement $Z = X - U$. Then the categories $\mathrm{Sh}(U)$, $\mathrm{Sh}(X)$ and $\mathrm{Sh}(Z)$ are related by the following functors:

$j_*\colon \mathrm{Sh}(U) \to \mathrm{Sh}(X)$	pushforward	(left exact);
$j_!\colon \mathrm{Sh}(U) \to \mathrm{Sh}(X)$	extension by 0	(exact);
$j^*\colon \mathrm{Sh}(X) \to \mathrm{Sh}(U)$	restriction	(exact);
$i_*\colon \mathrm{Sh}(Z) \to \mathrm{Sh}(X)$	pushforward	(exact);
$i^*\colon \mathrm{Sh}(X) \to \mathrm{Sh}(Z)$	restriction	(exact);
$i^!\colon \mathrm{Sh}(X) \to \mathrm{Sh}(Z)$	sections with support in Z	(left exact).

9.5. We have adjunctions:

$$(j_!, j^*, j_*) \qquad \text{and} \qquad (i^*, i_*, i^!),$$

where (x, y, z) means x is left adjoint to y and y is left adjoint to z.

9.6. Further, we have the identities:

$$j^* i_* = 0, \qquad i^* j_! = 0 \qquad \text{and} \qquad i^! j_* = 0.$$

9.7. The adjunction maps give canonical isomorphisms:

$$i^* i_* \xrightarrow{\sim} \mathrm{id} \xrightarrow{\sim} i^! i_* \qquad \text{and} \qquad j^* j_* \xrightarrow{\sim} \mathrm{id} \xrightarrow{\sim} j^* j_!.$$

9.8. The adjunction maps also give exact sequences

$$0 \to j_! j^* F \to F \to i_* i^* F \to 0 \qquad \text{and} \qquad 0 \to i_* i^! F \to F \to j_* j^* F,$$

for every $F \in \mathrm{Sh}(X)$.

10. Mayer-Vietoris sequences

10.1. **Exercise.** *Meditate on why the title of this section is what it is.*

10.2. Let $j\colon U \hookrightarrow X$ be the incluson of an open subspace and let $i\colon Z \hookrightarrow X$ be the inclusion of a closed subspace. Note: we are *not* assuming $Z = X - U$. Let $F \in \mathrm{Sh}(X)$. Define

$$\begin{aligned} \Gamma_U(F) &= j_* j^* F, \\ \Gamma_Z(F) &= i_* i^! F, \\ F_U &= j_! j^* F, \\ F_Z &= i_* i^* F. \end{aligned}$$

10.3. Now let $Z_1, Z_2 \subseteq X$ be closed subsets that cover X. Then one obtains a short exact sequence

$$0 \to F \xrightarrow{+} F_{Z_1} \oplus F_{Z_2} \xrightarrow{-} F_{Z_1 \cap Z_2} \to 0,$$

where $+ = \left(\begin{smallmatrix} \eta_1 \\ \eta_2 \end{smallmatrix}\right)$, $- = \left(\begin{smallmatrix} \eta'_1 & -\eta'_2 \end{smallmatrix}\right)$, and $\eta_i \colon F \to F_{Z_i}$, $\eta'_i \colon F_{Z_i} \to F_{Z_1 \cap Z_2}$, $i \in \{1, 2\}$, are the adjunction maps.

10.4. Similarly, if $U_1, U_2 \subseteq X$ are open subsets that cover X, then we have a short exact sequence

$$0 \to F_{U_1 \cap U_2} \xrightarrow{+} F_{U_1} \oplus F_{U_2} \xrightarrow{-} F \to 0,$$

where $+ = \left(\begin{smallmatrix} \varepsilon_1 & \varepsilon_2 \end{smallmatrix}\right)$, $- = \begin{pmatrix} \varepsilon'_1 \\ -\varepsilon'_2 \end{pmatrix}$, and $\varepsilon_i \colon F_{U_1 \cap U_2} \to F_{U_i}$, $\varepsilon'_i \colon F_{U_i} \to F$, $i \in \{1, 2\}$, are the adjunction maps.

10.5. Analogously, we have exact sequences

$$0 \to F \xrightarrow{+} \Gamma_{U_1}(F) \oplus \Gamma_{U_2}(F) \xrightarrow{-} \Gamma_{U_1 \cap U_2}(F)$$

and

$$0 \to \Gamma_{Z_1 \cap Z_2}(F) \xrightarrow{+} \Gamma_{Z_1}(F) \oplus \Gamma_{Z_2}(F) \xrightarrow{-} F,$$

where the maps $+$ and $-$ are induced by adjunction maps.

10.6. **Exercise.** *Verify that these sequences are indeed exact.* Hint: *This can be done extremely quickly if you look at stalks.*

11. Gluing sheaves

11.1. Let X be a space and let $\bigcup_{i \in I} U_i = X$ be an open cover of X. Set $U_{ij} = U_i \cap U_j$ and set $U_{ijk} = U_i \cap U_j \cap U_k$ for $i, j, k \in I$.

11.2. Consider a sheaf F on X. Set $F_i = F|_{U_i}$, for $i, j \in I$. Then we have canonical isomorphisms

$$\theta_i \colon F|_{U_{ij}} \xrightarrow{\sim} F_i|_{U_{ij}},$$

for all $i, j \in I$. Let $\theta_{ji} = \theta_j \circ \theta^{-1}$. Then

(i) $\theta_{ii} = \mathrm{id}_{F_i}$, for all $i \in I$;
(ii) $(\theta_{ij}|_{U_{ijk}}) \circ (\theta_{jk}|_{U_{ijk}}) = \theta_{ik}|_{U_{ijk}}$, for all $i, j, k \in I$.

In fact, one can reconstruct F from the above data:

11.3. **Proposition.** *Let $\bigcup_{i \in I} U_i = X$ be an open cover of X. Assume to be given the following data:*

(i) *a sheaf F_i on U_i for each $i \in I$;*
(ii) *for each pair $(i, j) \in I \times I$ an isomorphism $\theta_{ji} \colon F_i|_{U_{ij}} \xrightarrow{\sim} F_j|_{U_{ij}}$. These isomorphisms satisfying:*

$$\theta_{ii} = \mathrm{id}_{F_i} \quad \textit{and} \quad (\theta_{ij})|_{U_{ijk}} \circ (\theta_{jk})|_{U_{ijk}} = \theta_{ik}|_{U_{ijk}}$$

for all $i, j, k \in I$.

Then there exists a sheaf F on X, and isomorphisms $f_i \colon F|_{U_i} \xrightarrow{\sim} F_i$, such that $\theta_{ij} \circ f_j|_{U_{ij}} = f_i|_{U_{ij}}$. Moreover, the family $(F, \{f_i\}_{i \in I})$ is unique up to canonical isomorphism.

Proof sketch. For each open set U of X, define $F(U)$ to be the submodule of $\prod_{i \in I} F_i(U \cap U_i)$ consisting of families $(s_i)_{i \in I}$ such that for any $(i, j) \in I \times I$,

$$\theta_{ji}(s_i|_{U \cap U_{ji}}) = s_j|_{U \cap U_{ji}}.$$

This assignment gives a sheaf. The isomorphisms f_i are induced by the projection maps $\prod_j F_j(U \cap U_j) \to F_i(U \cap U_i)$.

Now suppose $(G, \{g_i\}_{i \in I})$ is another family satisfying the same properties. Then the isomorphisms

$$g_i^{-1} \circ f_i \colon F|_{U_i} \xrightarrow{\sim} G|_{U_i}$$

glue to give an isomorphism $F \xrightarrow{\sim} G$. □

11.4. **Example.** Let $U_1, U_2 \subseteq S^1$ be the complements of the north and south pole respectively. Let $U_{12}^{\pm}$ denote the two connected components of $U_1 \cap U_2$. Let $\zeta \in \mathbf{C}^\times$. Define a sheaf L_ζ on S^1 by gluing $\underline{\mathbf{C}}_{U_1}$ and $\underline{\mathbf{C}}_{U_2}$ as follows. For $\varepsilon \in \{+, -\}$, let

$$\theta_\varepsilon \colon \underline{\mathbf{C}}_{U_1}|_{U_{12}^\varepsilon} \to \underline{\mathbf{C}}_{U_2}|_{U_{12}^\varepsilon}$$

be defined by

$$\theta_+ = 1 \quad \text{and} \quad \theta_- = \zeta.$$

If $\zeta = 1$, then we just obtain the constant sheaf $\underline{\mathbf{C}}_{S^1}$. However, for $\zeta \neq 1$, the sheaf L_ζ is not isomorphic to $\underline{\mathbf{C}}_{S^1}$.

11.5. **Exercise.** *What is* $\Gamma(S^1; L_\zeta)$*?*

12. Additional exercises

12.1. **Exercise.** *Let $f \colon X \to Y$ be a map between spaces. Let G be a presheaf on Y. Construct a canonical isomorphism*

$$(f^t G)^\# \xrightarrow{\sim} f^{-1}(G^\#).$$

12.2. **Exercise.** *Let X be a space and let $\bigcup_i U_i = X$ be an open cover of X. Write $j_i \colon U_i \hookrightarrow X$ for the inclusion map. Let F be a presheaf on X. Suppose $j_i^t F$ is a sheaf for all i. Prove that F is a sheaf.*

12.3. **Exercise.** *Let $F \colon \mathcal{A} \to \mathcal{B}$ and $G \colon \mathcal{B} \to \mathcal{A}$ be functors between categories $\mathcal{A}$ and $\mathcal{B}$. Suppose F is left adjoint to G. Show that G preserves limits and F preserves colimits.*

12.4. **Exercise.** *Let $f \colon X \to Y$ be a map between spaces. In general, is it possible for f_* to have a right adjoint?*

12.5. **Exercise.** *Let F, G be sheaves on a space X. Show that the assignment*

$$U \mapsto \mathrm{Hom}_{\mathrm{Sh(U)}}(F|_U, G|_U)$$

defines a sheaf on X (what are the restriction maps?). This sheaf is usually denoted by $\mathscr{H}om(F, G)$ and is called sheaf Hom. *Note that* $\Gamma(X; \mathscr{H}om(F, G)) = \mathrm{Hom}(F, G)$.

12.6. **Exercise.** *Let*

$$X = \{(x, y) \in \mathbf{R}^2 \mid xy \geq 1\}.$$

Define

$$f \colon X \to \mathbf{R}, \quad (x, y) \mapsto y.$$

Describe $f_* \underline{\mathbf{C}}_X$ as explicitly as possible.

12.7. **Exercise.** *In this exercise all sheaves are understood to be sheaves of $\mathbf{C}$-vector spaces. Define*

$$f \colon S^1 \to S^1, \quad z \mapsto z^2, \quad \text{and} \quad g \colon S^1 \to S^1, \quad z \mapsto z^3.$$

The 'square root' sheaf on S^1 is defined to be

$$\mathcal{S} = \{a\phi \mid a \in \mathbf{C},\ \phi \colon S^1 \to S^1 \text{ is a continuous map such that } f\phi = \mathrm{id}\}.$$

The 'cube root' sheaf on S^1 is defined to be

$$\mathcal{Q} = \{a\phi \mid a \in \mathbf{C}, \phi \colon S^1 \to S^1 \text{ is a continuous map such that } g\phi = \mathrm{id}\}.$$

(i) *Describe* $\mathcal{S}$ *and* $\mathcal{Q}$ *as explicitly as possible.* Remark/hint: *See Example 11.4. Both* $\mathcal{S}$ *and* $\mathcal{Q}$ *are examples of what are called 'local systems' or 'twisted constant sheaves'. We have* $\mathcal{S}_x \simeq \mathcal{Q}_x \simeq (\underline{\mathbf{C}}_{S^1})_x \simeq \mathbf{C}$. *However, no two of these three sheaves are isomorphic to each other. Both* $\mathcal{S}$ *and* $\mathcal{Q}$ *have no global sections. To understand* $\mathcal{S}$ *you may want to pick an open cover of* S^1 *consisting of two sets such that* $\mathcal{S}$ *restricted to either of these sets is the constant sheaf. Now something interesting will happen with the restriction maps to the intersection of these sets: there will be a 'twist' that will prevent you from gluing sections to obtain a non-trivial global section. Your job is to figure out what this 'twist' is. Now do a similar thing for* $\mathcal{Q}$. *Can you make an educated guess for the general situation of 'n-th root' sheaves?*

(ii) *Describe the sheaves* $f_*\underline{\mathbf{C}}_{S^1}$, $g_*\underline{\mathbf{C}}_{S^1}$, $f_*\mathcal{S}$ *and* $g_*\mathcal{Q}$.

(iii) *Describe the sheaves* $f^{-1}\underline{\mathbf{C}}_{S^1}$ *and* $g^{-1}\underline{\mathbf{C}}_{S^1}$. Hint: *let* $h\colon X \to Y$ *be* any *map between spaces. What is* $h^{-1}\underline{\mathbf{C}}_Y$*?*

12.8. **Exercise.** *Let* $U_1, U_2 \subseteq S^1$ *be the complements of the north and south poles respectively. Classify (up to isomorphism) all sheaves* F *(of* $\mathbf{C}$*-vector spaces) on* S^1 *satisfying the following property:*

$$F|_{U_1} \simeq \underline{\mathbf{C}}_{U_1} \quad \text{and} \quad F|_{U_2} \simeq \underline{\mathbf{C}}_{U_2}.$$

Also describe $\Gamma(S^1; F)$ *for* F *as above. Now let* F_1, F_2 *be sheaves on* S^1 *satisfying the above. Describe* $\mathrm{Hom}(F_1, F_2)$. Hint: *Do the previous exercise.*

12.9. **Exercise.** *Define a space* $\mathbf{P}^1$ *as follows. As a set* $\mathbf{P}^1 = \mathbf{C} \sqcup \{\infty\}$. *Non-trivial closed sets in* $\mathbf{P}^1$ *are finite sets. So that non-empty open sets are of the form* $\mathbf{P}^1 - \{x_1, \ldots, x_n\}$. *Define a sheaf* $\mathcal{O}_{\mathbf{P}^1}$ *on* $\mathbf{P}^1$ *via the assignment*

$$U \mapsto \{\textit{rational functions } U \to \mathbf{C} \textit{ that have no poles in } U\},$$

where a rational function $g(x)$ *is said to have no pole at* ∞ *if and only if* $g(\frac{1}{y})$ *has no pole at* $y = 0$.

Compute:

(i) $\Gamma(\mathbf{P}^1 - \{0\}; \mathcal{O}_{\mathbf{P}^1})$;
(ii) $\Gamma(\mathbf{P}^1 - \{\infty\}; \mathcal{O}_{\mathbf{P}^1})$;
(iii) $\Gamma(\mathbf{P}^1; \mathcal{O}_{\mathbf{P}^1})$.

13. The adjunction (f^{-1}, f_*) revisited

13.1. Let $f\colon X \to Y$ be a map of spaces. I will now sketch another proof of the fact that f^{-1} is left adjoint to f_* that avoids unit/counit maps.

13.2. Write $\mathrm{Sh}(-)$ and $\mathrm{PreSh}(-)$ for the category of sheaves and presheaves on $-$, respectively. Let $\mathrm{Op}(-)$ denote the category of open sets of $-$.

13.3. Let $F \in \mathrm{Sh}(X)$ and let $G \in \mathrm{Sh}(Y)$. We need to construct a bifunctorial isomorphism

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}G, F) \simeq \mathrm{Hom}_{\mathrm{Sh}(Y)}(G, f_*F).$$

It suffices to construct a bifunctorial isomorphism

$$\mathrm{Hom}_{\mathrm{PreSh}(X)}(f^{t}G, F) \simeq \mathrm{Hom}_{\mathrm{PreSh}(Y)}(G, f_*F).$$

Now an element $\alpha \in \mathrm{Hom}_{\mathrm{PreSh}(Y)}(G, f_*F)$ consists of a family of maps

$$\{\alpha_V\colon G(V) \to F(f^{-1}(V))\}_{V \in \mathrm{Op}(Y)}$$

that are compatible with the restriction maps. Equivalently, this is a family of maps

$$\{\alpha_U\colon G(V) \to F(U)\}_{V \in \mathrm{Op}(Y), U \in \mathrm{Op}(X), U \subseteq f^{-1}(V)}$$

that are compatible with the restriction maps. Consequently, we obtain a family of maps

$$\{\tilde{\alpha}_U\colon \varinjlim_{V \supseteq f(U)} G(V) \to F(U)\}_{U \in \mathrm{Op}(X)}$$

that are compatible with restriction maps. This is precisely the data of an element in $\mathrm{Hom}_{\mathrm{PreSh}(X)}(f^{t}G, F)$. A moments thought should convince you that this correspondence gives the require bifunctorial isomorphism.

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