CHOW GROUPS AND EQUIVARIANT GEOMETRY

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Introduction. Throughout, 'variety' will mean 'separated scheme of finite type over Spec($\mathbf{C}$)'. A G-variety will mean a variety endowed with the action of a *linear* algebraic group G. The existence of functorial mixed Hodge structures on the rational cohomology, Borel-Moore homology, equivariant cohomology, etc., of a variety will be freely used (see [D]). For homology (Borel-Moore, equivariant, etc.), H_* will be called *pure* if each H_i is a pure Hodge structure of weight $-i$.

Theorem 1. *Let X be a variety on which a linear algebraic group acts with finitely many orbits. If the (Borel-Moore) homology $H_*(X;\mathbf{Q})$ is pure (for instance, if X is rationally smooth and complete), then the cycle class map*

$$CH_*(X)_{\mathbf{Q}} \xrightarrow{\sim} H_*(X;\mathbf{Q}),$$

from rational Chow groups to homology, is a degree doubling isomorphism.

This extends a result of Fulton-MacPherson-Sottile-Sturmfels [FMSS] from solvable groups to arbitrary linear algebraic groups. The price paid is that while most of the results of [FMSS] hold integrally, we deal exclusively with rational coefficients. Our arguments are quite different from those in [FMSS]. In particular, Theorem 1 is deduced from statements in the equivariant context.

For a G-variety X, write $A^G_*(X)_{\mathbf{Q}}$ for its rational equivariant Chow groups. Let $H^G_*(X;\mathbf{Q})$ denote the G-equivariant (Borel-Moore) homology of X, and let $W_\bullet$ be the weight filtration on $H^G_*(X;\mathbf{Q})$.

Theorem 2. *Let G be a linear algebraic group acting on a variety X. Assume X admits finitely many orbits. Then the cycle class map*

$$A^G_i(X)_{\mathbf{Q}} \xrightarrow{\sim} W_{-2i}H^G_{2i}(X;\mathbf{Q})$$

is an isomorphism for each $i \in \mathbf{Z}$.

This is established by mimicking B. Totaro's arguments from [T1]. Combined with Lemma 7, it yields the equivariant analogue of Theorem 1.

Corollary 3. *Let G be a linear algebraic group acting on a variety X. Assume X admits finitely many orbits, and that $H^G_*(X;\mathbf{Q})$ is pure. Then the cycle class map*

$$A^G_*(X)_{\mathbf{Q}} \xrightarrow{\sim} H^G_*(X;\mathbf{Q})$$

is a degree doubling isomorphism.

Now Theorem 1 follows via restriction from the equivariant to the non-equivariant context, using a result of M. Brion (Lemma 4).

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Examples

Flag varieties and related spaces. A straightforward illustration of Theorem 1 is provided by taking X to be a complete homogeneous space. In fact, in this situation the conclusions of Theorem 1 hold over $\mathbf{Z}$. This follows from the Bruhat decomposition and is significantly more elementary than the arguments used in the current note. A slightly more involved example is that of a Bott-Samelson resolution of a Schubert variety (see [Gau]).

Toric and spherical varieties. Other examples following a similar pattern are provided by toric varieties, and more generally, by spherical varieties. These cases use elementary methods which essentially have two pieces of content. The first being that the variety in question is smooth and complete. The other being that the homology of the variety is generated by algebraic cycles - shown using explicit constructions (for instance, in the case of toric varieties, consider closures of the torus orbits).

Towards genuinely new examples. In all of the above examples, apart from wrapping everything into one general result and avoiding explicit constructions, Theorem 1 does not provide any new information and is in many ways overkill. The most interesting situations in which something new is brought to the table is when $H_*(X;\mathbf{Q})$ is pure even though X is not known to be rationally smooth, and no obvious stratification of X by vector spaces and/or algebraic tori is apparent. Here is a general situation, occuring frequently in representation theory, that yields many such examples.

Contracting slices. Let Y be a G-variety. A *contracting slice* at a point $y \in Y$ is the data of a locally closed subvariety $S \subset Y$ containing y, and satisfying:

(i) the map $G \times S \to Y$, $(g, y) \mapsto gy$ is smooth;
(ii) there exists a one parameter subgroup $\mathbf{C}^\times \to G$ that leaves S stable and contracts S to y.

Say that the G-action on Y *admits contracting slices* if each G-orbit contains a point that admits a contracting slice.

The significance of this notion is that if $\pi\colon E \to Y$ is a proper equivariant morphism, then $H_*(\pi^{-1}(y);\mathbf{Q})$ is known to be pure for every point $y \in Y$ which admits a contracting slice. A convenient, but by no means original, discussion explaining how this is equivalent to well known pointwise purity results for perverse sheaves (as in [BB, §5.2], [KL], [MS, §2.3], [S]) is contained in [V, §6]. In particular, if Y admits contracting slices and E is a G-variety admitting finitely many orbits, then Theorem 1 applies to $X = \pi^{-1}(y)$ for *any* proper equivariant morphism $\pi\colon E \to Y$ and $y \in Y$. Note that $\pi^{-1}(y)$ may not be rationally smooth (however, roughly speaking, the existence of contracting slices ensures that $\pi^{-1}(y)$ is an algebraic homotopy retract of a smooth variety).

Fibres of resolutions. A specific illustration of this, generalizing the example of Bott-Samelson resolutions, is as follows. Let G be a reductive group, $B \subset G$ a Borel subgroup, and G/B the corresponding flag variety. It is known that the B-action on G/B admits contracting slices (for instance, see the Appendix of [KL]). Let $Y \subset G/B$ be a Schubert variety. Let $\pi\colon E \to Y$ be a B-equivariant resolution of singularities. Then $H_*(\pi^{-1}(y);\mathbf{Q})$ is pure, by the above observations, for all $y \in Y$.

If B acts on E with finitely many orbits, then the action of the isotropy group $B_y \subset B$ on $\pi^{-1}(y)$ must also admit finitely orbits. Thus, under these assumptions, we are in the situation of Theorem 1 and it follows that $H_*(\pi^{-1}(y); \mathbf{Q})$ is generated by algebraic cycles.

Fibres of resolutions of toric varieties, spherical varieties, etc., provide several more such examples. In all of these situations Theorem 1 also immediately yields the parity vanishing phenomenon for stalks of intersection cohomology sheaves that is ubiquitous in representation theory (for instance, see [KL], [MS]).

A "non-example". Let X be a Springer fibre ([CG, Chapter 3] is a convenient reference). Then X occurs as the fibre of a proper equivariant morphism to the nilpotent cone (the Springer resolution). The nilpotent cone is known to admit contracting slices (for instance, see [CG, §3.7.14]). However, X does not necessarily satisfy the finite orbits assumption of Theorem 1. Regardless, the conclusions of Theorem 1 are known to hold (even over $\mathbf{Z}$). The argument involves type by type considerations and explicit constructions of appropriate stratifications (see [DPL]).

Proofs

Preliminaries. Let X be a G-variety. Write $\bar{H}^*_G(X; \mathbf{Q}(j))$ for the equivariant motivic cohomology of X (with 'coefficients' in $\mathbf{Q}(j)$). Write $\bar{H}^G_*(X; \mathbf{Q}(j))$ for the equivariant motivic (Borel-Moore) homology of X. In terms of the higher equivariant Chow groups $A^G_p(X, k)$ of [EG]:

$$\bar{H}^G_i(X; \mathbf{Q}(j)) = A^G_j(X, i-2j) \otimes \mathbf{Q}.$$

In particular, $\bar{H}^G_{2i}(X; \mathbf{Q}(i)) = A^G_i(X)_{\mathbf{Q}}$. It will be notationally convenient to set

$$A^i_G = \bar{H}^{2i}_G(\mathrm{Spec}(\mathbf{C}); \mathbf{Q}(i)).$$

Given a group morphism $H \to G$, there is a restriction map $A^*_G \to A^*_H$. There are analogous restriction maps for motivic homology.

Lemma 4. *If G is connected, then restriction induces an isomorphism:*

$$\mathbf{Q} \otimes_{A^*_G} A^G_*(X)_{\mathbf{Q}} \xrightarrow{\sim} CH_*(X)_{\mathbf{Q}}.$$

Proof. If G is reductive, then this is [B, Corollary 6.7(i)]. In general, if $U \subset G$ is the unipotent radical, then G/U is reductive, and restriction yields an isomorphism $A^*_{G/U} \xrightarrow{\sim} A^*_G$. Similarly for motivic homology (see [T2, Lemma 2.18]). □

There is a natural map $\bar{H}^G_i(X; \mathbf{Q}(j)) \to W_{-2j} H^G_i(X; \mathbf{Q})$. See [T1, §4] for a cogent explanation of this. The map

$$\bar{H}^G_{2i}(X; \mathbf{Q}(i)) = A^G_i(X)_{\mathbf{Q}} \to H^G_{2i}(X; \mathbf{Q})$$

is the *cycle class map*.

Weak property. A G-variety X satisfies the *weak property* if the cycle class map

$$\bar{H}^G_{2i}(X; \mathbf{Q}(i)) = A^G_i(X)_{\mathbf{Q}} \to W_{-2i} H^G_{2i}(X; \mathbf{Q})$$

is an isomorphism for each $i \in \mathbf{Z}$.

Strong property. A G-variety X satisfies the *strong property* if it satisfies the weak property and the map

$$\bar{H}^G_{2i+1}(X;\mathbf{Q}(i)) \to \mathrm{gr}^W_{-2i} H^G_{2i+1}(X;\mathbf{Q})$$

is surjective for each $i \in \mathbf{Z}$. Here $\mathrm{gr}^W_\bullet$ denotes the associated graded with respect to the weight filtration $W_\bullet$.

Lemma 5. *Let G be a linear algebraic group, and let $K \subset G$ be a closed subgroup. Then G/K satisfies the strong property (as a G-variety).*

Proof. The map $A^G_*(G/K)_\mathbf{Q} \xrightarrow{\sim} H^G_*(G/K;\mathbf{Q})$ is a degree doubling isomorphism (see [T2, Theorem 2.14]). □

Lemma 6. *Let X be a G-variety, $Z \subset X$ a G-stable closed subvariety, and $U = X - Z$ the open complement. If U satisfies the strong property and Z the weak, then X satisfies the weak property.*

Proof. We have a morphism of long exact sequences:

$$\begin{array}{ccccccccc} \bar{H}^G_{2i+1}(U;\mathbf{Q}(i)) & \to & \bar{H}^G_{2i}(Z;\mathbf{Q}(i)) & \to & \bar{H}^G_{2i}(X;\mathbf{Q}(i)) & \to & \bar{H}^G_{2i}(U;\mathbf{Q}(i)) & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \mathrm{gr}^W_{-2i} H^G_{2i+1}(U) & \longrightarrow & W_{-2i} H^G_{2i}(Z) & \longrightarrow & W_{-2i} H^G_{2i}(X) & \longrightarrow & W_{-2i} H^G_{2i}(U) & \longrightarrow & 0 \end{array}$$

where '$\mathbf{Q}$' has been omitted from the notation in the bottom row due to typesetting considerations. The first vertical map is surjective (strong property for U). The second and fourth vertical maps are isomorphisms by the weak property for Z and U respectively. So the third vertical map must also be an isomorphism. □

Proof of Theorem 2. Combine Lemma 5 and Lemma 6.

Proof of Corollary 3. Combine Theorem 2 with the following observation.

Lemma 7. *Let X be a variety on which a linear algebraic group G acts with finitely many orbits. Then the (Borel-Moore) homology $H_*(X;\mathbf{Q})$ is a sucessive extension of Hodge structures of type (n,n).*

Proof. We may assume $X = G/K$, where $K \subset G$ is a closed subgroup. Now $H^*(G/K;\mathbf{Q})$ is the K-equivariant cohomology of G. Consider the usual simplicial variety computing this (see [D, §6]). Filtering by skeleta yields a spectral sequence whose E_1 entries are of the form $H^q(K^{\times p} \times G;\mathbf{Q})$ [D, Proposition 8.3.5]. Now recall that the cohomology of a linear algebraic group is of type (n,n) [D, §9.1]. □

Proof of Theorem 1. We may assume that G is connected. Let H^*_G denote the equivariant cohomology ring of a point. Purity and Lemma 7 imply that $H_*(X;\mathbf{Q})$ is concentrated in even degrees. Purity also implies that the natural map

$$\mathbf{Q} \otimes_{H^*_G} H^G_*(X;\mathbf{Q}) \xrightarrow{\sim} H_*(X;\mathbf{Q})$$

is an isomorphism. Further, the cycle class map $A^G_*(X)_\mathbf{Q} \xrightarrow{\sim} H^G_*(X;\mathbf{Q})$ is an isomorphism by Corollary 3, since purity of $H_*(X;\mathbf{Q})$ implies purity of $H^G_*(X;\mathbf{Q})$.

Thus, combined with Lemma 4, we obtain a commutative diagram:

$$\begin{array}{ccccc} A_*^G(X)_{\mathbf{Q}} & \longrightarrow & \mathbf{Q}\otimes_{A_G^*} A_*^G(X)_{\mathbf{Q}} & \xrightarrow{\sim} & CH_*(X)_{\mathbf{Q}} \\ \sim\big\downarrow & & \big\downarrow & & \big\downarrow \\ H_*^G(X;\mathbf{Q}) & \longrightarrow & \mathbf{Q}\otimes_{H_G^*} H_*^G(X;\mathbf{Q}) & \xrightarrow{\sim} & H_*(X;\mathbf{Q}) \end{array}$$

Consequently, $CH_*(X)_{\mathbf{Q}} \xrightarrow{\sim} H_*(X;\mathbf{Q})$ is an isomorphism.

References

[BB] A. Beilinson, J. Bernstein, *A proof of Jantzen conjectures*, Advances in Soviet Math. **16** part I (1993), 1-50.

[B] M. Brion, *Equivariant Chow groups for torus actions*, Transformation Groups **2** (1997), 225-267.

[CG] N. Chriss, V. Ginzburg, *Representation Theory and Complex Geometry*, Birkhäuser (1997).

[DPL] C. DeConcini, C. Procesi, G. Lusztig, *Homology of the zero set of a nilpotent vector field on a flag manifold*, J.A.M.S. **1** (1988), 15-34.

[D] P. Deligne, *Théorie de Hodge III*, Publ. IHÉS **44** (1974), 5-77.

[EG] D. Edidin, W. Graham, *Equivariant intersection theory*, Invent. Math. **131** (1998), 595-634.

[FMSS] W. Fulton, R. MacPherson, F. Sottile, B. Sturmfels, *Intersection theory on spherical varieties*, J. Alg. Geom. **4** (1995), 181-193.

[Gau] S. Gaussent, *The fibre of the Bott-Samelson resolution*, Indag. Math. (N.S.) **12** (2001), No. 4, 453-468.

[KL] G. Lusztig, D. Kazhdan, *Schubert varieties and Poincaré duality*, in Geometry of the Laplace operator, Proc. of Symposia in Pure Math XXXVI, AMS (1980).

[MS] J.G.M. Mars, T.A. Springer, *Hecke algebra representations related to spherical varieties*, Rep. Theory **2** (1998), 33-69.

[S] T.A. Springer, *A purity result for fixed point varieties in flag manifolds*, J. Fac. Sci., Univ. Tokyo, Sect. 1A **31** (1984), 231-238.

[T1] B. Totaro, *Chow groups, Chow cohomology, and linear varieties*, Forum of Math., Sigma 2 (2014), e17.

[T2] B. Totaro, *Group cohomology and Algebraic Cycles*, Cambridge Tracts in Math. **204** (2014).

[V] R. Virk, *Tate classes, equivariant geometry and purity*, arXiv:1507.07279.

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