

Affine and degenerate affine BMW algebras: Actions on tensor space

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Abstract

The affine and degenerate affine Birman-Murakami-Wenzl (BMW) algebras arise naturally in the context of Schur-Weyl duality for orthogonal and symplectic quantum groups and Lie algebras, respectively. Cyclotomic BMW algebras, affine and cyclotomic Hecke algebras, and their degenerate versions are quotients of the affine and degenerate affine BMW algebras. In this paper we explain how the affine and degenerate affine BMW algebras are tentalizers (tensor power centralizer algebras) by defining actions of the affine braid group and the degenerate affine braid algebra on tensor space and showing that, in important cases, these actions induce actions of the affine and degenerate affine BMW algebras. We then exploit the connection to quantum groups and Lie algebras to determine universal parameters for the affine and degenerate affine BMW algebras. Finally, we show that the universal parameters are central elements—the higher Casimir elements for orthogonal and symplectic enveloping algebras and quantum groups.

Introduction

This paper is a continuation of our study of the affine Birman-Murakami-Wenzl (BMW) algebras W_k and their degenerate versions \mathcal{W}_k . In [6] we defined the algebras W_k and \mathcal{W}_k and determined their centers. Each of these algebras contains a commuting family of “Jucys-Murphy elements”; following Nazarov and Beliakova-Blanchet [21, 3] we derived generating function formulas for specific central elements $z_k^{(\ell)} \in \mathcal{W}_k$ and $Z_k^{(\ell)} \in W_k$ in terms of the Jucys-Murphy elements.

In this paper we show that the algebras W_k and \mathcal{W}_k have a natural action on tensor space which provides a Schur-Weyl duality with the quantum group and enveloping algebras of classical type Lie algebras. In particular, the algebras W_k and \mathcal{W}_k arise in orthogonal and symplectic type, though we treat all the classical type cases uniformly. In complete analogy with the fact that affine BMW algebra is a quotient of the group algebra CB_k (over a commutative ring C) of the affine braid group of type A, the degenerate affine BMW algebra is a quotient of the *degenerate affine braid algebra* \mathcal{B}_k which we first defined in [6]. This analogy extends to the Schur-Weyl duality. In Theorem 1.2, we show that there is a natural action of \mathcal{B}_k on tensor space which commutes with an *arbitrary* finite-dimensional complex reductive Lie algebra \mathfrak{g} . In Theorem 3.3 we show that, when \mathfrak{g} is of classical type and the tensor space is constructed from the n -dimensional defining representation, the action of \mathcal{B}_k becomes an action of familiar algebras: the degenerate affine BMW algebra arises when $\mathfrak{g} = \mathfrak{so}_n$ or \mathfrak{sp}_n , and the degenerate affine Hecke algebra arises when $\mathfrak{g} = \mathfrak{gl}_n$ or \mathfrak{sl}_n .

The affine and degenerate affine BMW algebras depend on the choice of an infinite number of parameters. This is analogous to the way that the Iwahori-Hecke algebra depends on one parameter, often called q . Unfortunately, the infinite collection of parameters for the BMW algebras is not free; significant work has been done on when a collection is admissible [1, 28, 29, 9, 10, 12, 13, 14, 30]. In this work, we take a different point of view and produce *universal* parameters for the affine and degenerate affine BMW algebras. These universal parameters are symmetric functions which satisfy the admissibility conditions. In future work, we hope to show via representation theory that every choice of admissible complex parameters is a specialization of our universal parameters.

To compute the symmetric functions which arise as universal parameters, we use the Schur-Weyl duality to naturally identify them as elements of the center of the corresponding symplectic or orthogonal enveloping algebra or quantum group (which, by the Harish-Chandra isomorphism, is isomorphic to a ring of symmetric functions). Specifically, in Theorem 3.3 and Theorem 3.5 we execute computations which push the recursive formulas of Nazarov [21] and Beliakova-Blanchet [3] to the other side of the Schur-Weyl duality. This produces explicit formulas for the Harish-Chandra images of the corresponding central elements $z_V^{(\ell)}$ and $Z_V^{(\ell)}$ of the orthogonal and symplectic enveloping algebras and quantum groups. These computations are related to the calculations in, for example, [21], [19, Ch. 7] and [20].

In Section 4 we show that the central elements $z_V^{(\ell)}$ and $Z_V^{(\ell)}$ are the natural higher Casimir elements for orthogonal and symplectic enveloping algebras and quantum groups. In fact, we are able to show that the universal parameters $z_V^{(\ell)}$ of the degenerate affine BMW algebras coincide exactly with the higher Casimirs for orthogonal and symplectic Lie algebras given by Perelomov-Popov [23, 24]. Expositions of the Perelomov-Popov results are also found in [31,

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§127] and [19, §7.1]. Another computation shows that the universal parameters $Z_V^{(\ell)}$ of the affine BMW algebras coincide with the central elements in quantum groups defined by Reshetikhin-Takhtajan-Faddeev central elements defined in [25, Theorem 14]. To execute our computation we have relied on a remarkable identity of Baumann [2, Theorem 1].

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1 Actions of general type tentalizers

Our goal in Section 2 is to provide a tensor space action of the affine Birman-Murakami-Wenzl (BMW) algebra W_k and its degenerate version \mathcal{W}_k by way of the group algebra of the affine braid group CB_k and the degenerate affine braid algebra \mathcal{B}_k , respectively. The definition of the degenerate affine braid algebra \mathcal{B}_k makes the Schur-Weyl duality framework completely analogous in both the affine and degenerate affine cases.

In this section, we define CB_k and \mathcal{B}_k and show that both act on tensor space of the form $M \otimes V^{\otimes k}$. In the degenerate affine case this action commutes with a complex reductive Lie algebra \mathfrak{g} , and the ground ring C is the center of $U\mathfrak{g}$. In the affine case this action commutes with the Drinfeld-Jimbo quantum group $U_h\mathfrak{g}$, and the ground ring C is the center of $U_h\mathfrak{g}$. As we will see in Section 2, the affine and degenerate affine BMW algebras arise when \mathfrak{g} is orthogonal or symplectic and V is the defining representation; similarly, the degenerate affine Hecke algebras arise when \mathfrak{g} is of type \mathfrak{gl}_n or \mathfrak{sl}_n and V is the defining representation. In the case when M is the trivial representation and \mathfrak{g} is \mathfrak{so}_n , the elements y_1, \dots, y_k in \mathcal{B}_k become the Jucys-Murphy elements for the Brauer algebras used in [21]; in the case that $\mathfrak{g} = \mathfrak{gl}_n$, these become the classical Jucys-Murphy elements in the group algebra of the symmetric group.

The action of the degenerate affine braid algebra \mathcal{B}_k and the action of the affine braid group B_k on $M \otimes V^{\otimes k}$ each provide *Schur functors*

$$F_V^\lambda: \{U\text{-modules}\} \longrightarrow \begin{cases} \{\mathcal{B}_k\text{-modules}\}, & \text{if } U = U\mathfrak{g}, \\ \{B_k\text{-modules}\}, & \text{if } U = U_h\mathfrak{g}. \end{cases} \quad (1.1)$$

$$M \longmapsto \text{Hom}_U(M(\lambda), M \otimes V^{\otimes k})$$

where in each case $\text{Hom}_U(M(\lambda), M \otimes V^{\otimes k})$ is the vector space of highest weight vectors of weight λ in $M \otimes V^{\otimes k}$ (since $M(\lambda)$ denotes the Verma module of highest weight λ). These ubiquitous functors transfer representation theoretic information back and forth either between $U\mathfrak{g}$ and \mathcal{B}_k or between $U_h\mathfrak{g}$ and CB_k .

1.1 The degenerate affine braid algebra action

Let C be a commutative ring and let S_k denote the symmetric group on $\{1, \dots, k\}$. For $i \in \{1, \dots, k\}$, write s_i for the transposition in S_k that switches i and $i + 1$. The *degenerate affine braid algebra* is the algebra \mathcal{B}_k over C generated by

$$t_u \quad (u \in S_k), \quad \kappa_0, \kappa_1, \quad \text{and} \quad y_1, \dots, y_k, \quad (1.2)$$

with relations

$$t_u t_v = t_{uv}, \quad y_i y_j = y_j y_i, \quad \kappa_0 \kappa_1 = \kappa_1 \kappa_0, \quad \kappa_0 y_i = y_i \kappa_0, \quad \kappa_1 y_i = y_i \kappa_1, \quad (1.3)$$

$$\kappa_0 t_{s_i} = t_{s_i} \kappa_0, \quad \kappa_1 t_{s_1} \kappa_1 t_{s_1} = t_{s_1} \kappa_1 t_{s_1} \kappa_1, \quad \text{and} \quad \kappa_1 t_{s_j} = t_{s_j} \kappa_1, \quad \text{for } j \neq 1, \quad (1.4)$$

$$t_{s_i} (y_i + y_{i+1}) = (y_i + y_{i+1}) t_{s_i}, \quad \text{and} \quad y_j t_{s_i} = t_{s_i} y_j \quad \text{for } j \neq i, i+1, \quad (1.5)$$

$$\kappa_1 t_{s_1} y_1 t_{s_1} = t_{s_1} y_1 t_{s_1} \kappa_1, \quad (1.6)$$

and, for $i = 1, \dots, k-2$,

$$t_{s_i} t_{s_{i+1}} \gamma_{i,i+1} t_{s_{i+1}} t_{s_i} = \gamma_{i+1,i+2}, \quad \text{where} \quad \gamma_{i,i+1} = y_{i+1} - t_{s_i} y_i t_{s_i}. \quad (1.7)$$

This presentation highlights the ‘‘Jucys-Murphy’’ elements y_1, \dots, y_k for the degenerate affine BMW algebra \mathcal{W}_k as in [21]. However, the algebra \mathcal{B}_k also admits the following presentation, which highlights its natural action on tensor space (as we will see in Theorem 1.2).

Theorem 1.1. [6, Theorem 2.1] *The degenerate affine braid algebra \mathcal{B}_k has another presentation by generators*

$$t_u \quad (u \in S_k), \quad \kappa_0, \dots, \kappa_k \quad \text{and} \quad \gamma_{i,j}, \quad \text{for } 0 \leq i, j \leq k \text{ with } i \neq j, \quad (1.8)$$

and relations

$$t_u t_v = t_{uv}, \quad t_w \kappa_i t_w^{-1} = \kappa_{w(i)}, \quad t_w \gamma_{i,j} t_w^{-1} = \gamma_{w(i),w(j)}, \quad (1.9)$$

$$\kappa_i \kappa_j = \kappa_j \kappa_i, \quad \kappa_i \gamma_{\ell,m} = \gamma_{\ell,m} \kappa_i, \quad (1.10)$$

$$\gamma_{i,j} = \gamma_{j,i}, \quad \gamma_{p,r} \gamma_{\ell,m} = \gamma_{\ell,m} \gamma_{p,r}, \quad \text{and} \quad \gamma_{i,j} (\gamma_{i,r} + \gamma_{j,r}) = (\gamma_{i,r} + \gamma_{j,r}) \gamma_{i,j}, \quad (1.11)$$

for $p \neq \ell$ and $p \neq m$ and $r \neq \ell$ and $r \neq m$ and $i \neq j$, $i \neq r$ and $j \neq r$.

The conversion between the two presentations is given by the formulas $\kappa_0 = \kappa_0$, $\kappa_1 = \kappa_1$, $t_w = t_w$,

$$y_j = \frac{1}{2} \kappa_j + \sum_{0 \leq \ell < j} \gamma_{\ell,j}, \quad (1.12)$$

and the formulas in (1.9). Set

$$c_0 = \kappa_0 \quad \text{and} \quad c_j = \kappa_0 + 2(y_1 + \dots + y_j), \quad (1.13)$$

for $j = 1, 2, \dots, k$. Then c_0, \dots, c_k pairwise commute,

$$y_j = \frac{1}{2} (c_j - c_{j-1}) \quad \text{and} \quad c_j = \sum_{i=0}^j \kappa_i + 2 \sum_{0 \leq \ell < m \leq j} \gamma_{\ell,m}. \quad (1.14)$$

Let \mathfrak{g} be a finite-dimensional complex Lie algebra with a symmetric nondegenerate ad-invariant bilinear form i.e.,

$$\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{C}, \quad \text{with} \quad \langle [x_1, x_2], x_3 \rangle + \langle x_2, [x_1, x_3] \rangle = 0$$

and $\langle x_1, x_2 \rangle = \langle x_2, x_1 \rangle$, for $x_1, x_2, x_3 \in \mathfrak{g}$. Let $B = \{b_1, \dots, b_n\}$ be a basis for \mathfrak{g} and let $\{b_1^*, \dots, b_n^*\}$ be the dual basis with respect to $\langle \cdot, \cdot \rangle$. The Casimir is

$$\kappa = b_1 b_1^* + \dots + b_n b_n^* = \sum_{b \in B} b b^* \quad \text{and} \quad \kappa \in Z(U\mathfrak{g}), \quad (1.15)$$

where $Z(U\mathfrak{g})$ is the center of the enveloping algebra $U\mathfrak{g}$ (see [4, I §3 Prop. 11]). Since the coproduct on \mathfrak{g} is defined by $\Delta(x) = x \otimes 1 + 1 \otimes x$ for $x \in \mathfrak{g}$,

$$\Delta(\kappa) = \kappa \otimes 1 + 1 \otimes \kappa + 2\gamma, \quad \text{where} \quad \gamma = \sum_{b \in B} b \otimes b^*. \quad (1.16)$$

Theorem 1.2. *Let \mathfrak{g} be a finite-dimensional complex Lie algebra with a symmetric nondegenerate ad-invariant bilinear form $\langle \cdot, \cdot \rangle$, and let $U\mathfrak{g}$ be the universal enveloping algebra. Let $C = Z(U\mathfrak{g})$ be the center of $U\mathfrak{g}$, κ be the Casimir in C , and $\gamma = \sum_b b \otimes b^*$ as in (1.16). Let M and V be \mathfrak{g} -modules and let $s_1 \cdot (u \otimes v) = v \otimes u$, for $u, v \in V$.*

The degenerate affine braid algebra \mathcal{B}_k acts on $M \otimes V^{\otimes k}$ via $\Phi : \mathcal{B}_k \rightarrow \text{End}(M \otimes V^{\otimes k})$ defined by

$$\begin{aligned} \Phi(t_{s_i}) &= \text{id}_M \otimes \text{id}_V^{\otimes i-1} \otimes s_1 \otimes \text{id}_V^{\otimes k-i-1} \text{ so that } S_k \text{ acts by permuting tensor factors of } V, \\ \Phi(\kappa_i) &\text{ is } \kappa \text{ acting in the } i\text{th factor of } V \text{ in } M \otimes V^{\otimes k} \text{ and } \Phi(\kappa_0) \text{ is } \kappa \text{ acting on } M, \\ \Phi(\gamma_{\ell,m}) &\text{ is } \gamma \text{ acting in the } \ell\text{th and } m\text{th factors of } V \text{ in } M \otimes V^{\otimes k}, \\ \Phi(\gamma_{0,\ell}) &\text{ is } \gamma \text{ acting on } M \text{ and the } \ell\text{th factor of } V \text{ in } M \otimes V^{\otimes k}, \\ \Phi(c_i) &\text{ is } \kappa \text{ acting on } M \text{ and the first } i \text{ factors of } V, \\ \Phi(z) &= z \otimes \text{id}_V^{\otimes k} \text{ for } z \in C. \end{aligned}$$

This action of \mathcal{B}_k commutes with the $U\mathfrak{g}$ -action on $M \otimes V^{\otimes k}$.

Proof. Since $\kappa \in Z(U\mathfrak{g})$ the operators $\Phi(\kappa_i)$ are in $\text{End}_{\mathfrak{g}}(M \otimes V^{\otimes k})$. From the relation $2\gamma = \Delta(\kappa) - \kappa \otimes 1 - 1 \otimes \kappa$ it also follows that $\Phi(\gamma_{\ell,m}) \in \text{End}_{\mathfrak{g}}(M \otimes V^{\otimes k})$.

All of the relations in Theorem 1.1 except the last relations in (1.11) follow from consideration of the tensor factors being acted upon. The last relations in (1.11) are established by the computation

$$\begin{aligned} \gamma_{1,2}(\gamma_{1,3} + \gamma_{2,3})(v \otimes w \otimes z) &= \gamma_{1,2} \left(\sum_i b_i v \otimes w \otimes b_i^* z + v \otimes b_i w \otimes b_i^* z \right) \\ &= \gamma_{1,2} \left(\sum_i \Delta(b_i) \otimes b_i^* \right) (v \otimes w \otimes z) = \left(\sum_i \Delta(b_i) \otimes b_i^* \right) \gamma_{1,2}(v \otimes w \otimes z) \\ &= (\gamma_{1,3} + \gamma_{2,3})\gamma_{1,2}(v \otimes w \otimes z), \end{aligned}$$

for $v, w, z \in V$. Recursively applying the coproduct formula from (1.16) connects the action of c_i with the action of κ_i and $\gamma_{\ell,m}$ as in the second formula in (1.14). \square

For a $U\mathfrak{g}$ -module M let

$$\begin{array}{ccc} \kappa_M: & M & \longrightarrow & M \\ & m & \longmapsto & \kappa m \end{array} \quad \text{where } \kappa \text{ is the Casimir}$$

as in (1.15). If M is a $U\mathfrak{g}$ -module generated by a highest weight vector v_λ^+ of weight λ then

$$\kappa_M = \langle \lambda, \lambda + 2\rho \rangle \text{id}_M, \quad \text{where } \rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha \quad (1.17)$$

is the half-sum of the positive roots (see [4, VIII §2 no. 3 Prop. 6 and VIII §6 no. 4 Cor. to Prop. 7]). By equation (1.16), if $M = L(\mu)$ and $N = L(\nu)$ are finite-dimensional irreducible $U\mathfrak{g}$ -modules of highest weights μ and ν respectively, then γ acts on the $L(\lambda)$ -isotypic component of the decomposition $L(\mu) \otimes L(\nu) \cong \bigoplus_\lambda L(\lambda)^{\oplus c_{\mu\nu}^\lambda}$ by the constant

$$\frac{1}{2} (\langle \lambda, \lambda + 2\rho \rangle - \langle \mu, \mu + 2\rho \rangle - \langle \nu, \nu + 2\rho \rangle). \quad (1.18)$$

is a U -module isomorphism. The quasitriangularity of a ribbon Hopf algebra provides the braid relation (see, for example, [22, (2.12)]),

$$(\check{R}_{MN} \otimes \text{id}_P)(\text{id}_N \otimes \check{R}_{MP})(\check{R}_{NP} \otimes \text{id}_M) = (\text{id}_M \otimes \check{R}_{NP})(\check{R}_{MP} \otimes \text{id}_N)(\text{id}_P \otimes \check{R}_{MN}).$$

Theorem 1.3. *Let \mathfrak{g} be a finite-dimensional complex Lie algebra with a symmetric nondegenerate ad-invariant bilinear form, let $U = U_h\mathfrak{g}$ be the corresponding Drinfeld-Jimbo quantum group and let $C = Z(U)$ be the center of $U_h\mathfrak{g}$. Let M and V be U -modules. Then $M \otimes V^{\otimes k}$ is a CB_k -module with action given by*

$$\begin{aligned} \Phi: \quad CB_k &\longrightarrow \text{End}_U(M \otimes V^{\otimes k}) \\ T_i &\longmapsto \check{R}_i, & 1 \leq i \leq k-1, \\ X^{\varepsilon_1} &\longmapsto \check{R}_0^2, \\ z &\longmapsto z_M, \end{aligned} \quad (1.26)$$

where $z_M = z \otimes \text{id}_V^{\otimes k}$,

$$\check{R}_i = \text{id}_M \otimes \text{id}_V^{\otimes(i-1)} \otimes \check{R}_{VV} \otimes \text{id}_V^{\otimes(k-i-1)} \quad \text{and} \quad \check{R}_0^2 = (\check{R}_{MV}\check{R}_{VM}) \otimes \text{id}_V^{\otimes(k-1)},$$

with \check{R}_{MV} as in (1.25). The CB_k action commutes with the U -action on $M \otimes V^{\otimes k}$.

Proof. The relations (1.19) and (1.22) are consequences of the definition of the action of T_i and X^{ε_1} . The relations (1.20) and (1.21) follow from the following computations:

$$\check{R}_i\check{R}_{i+1}\check{R}_i = \begin{array}{c} | \quad | \\ \diagdown \quad \diagup \\ | \quad | \\ \diagup \quad \diagdown \\ | \quad | \end{array} = \begin{array}{c} | \quad | \\ \diagup \quad \diagdown \\ | \quad | \\ \diagdown \quad \diagup \\ | \quad | \end{array} = \check{R}_{i+1}\check{R}_i\check{R}_{i+1}$$

and

$$\check{R}_0^2\check{R}_1\check{R}_0^2\check{R}_1 = \begin{array}{c} | \quad | \\ \diagdown \quad \diagup \\ | \quad | \\ \diagup \quad \diagdown \\ | \quad | \\ \diagdown \quad \diagup \\ | \quad | \\ \diagup \quad \diagdown \\ | \quad | \end{array} = \begin{array}{c} | \quad | \\ \diagup \quad \diagdown \\ | \quad | \\ \diagdown \quad \diagup \\ | \quad | \\ \diagup \quad \diagdown \\ | \quad | \\ \diagdown \quad \diagup \\ | \quad | \end{array} = \begin{array}{c} | \quad | \\ \diagdown \quad \diagup \\ | \quad | \\ \diagup \quad \diagdown \\ | \quad | \\ \diagdown \quad \diagup \\ | \quad | \\ \diagup \quad \diagdown \\ | \quad | \end{array} = \begin{array}{c} | \quad | \\ \diagup \quad \diagdown \\ | \quad | \\ \diagdown \quad \diagup \\ | \quad | \\ \diagup \quad \diagdown \\ | \quad | \\ \diagdown \quad \diagup \\ | \quad | \end{array} = \check{R}_1\check{R}_0^2\check{R}_1\check{R}_0^2.$$

□

Let $v = e^{-h\rho}u$ be the ribbon element in $U = U_h\mathfrak{g}$. For a $U_h\mathfrak{g}$ -module M define

$$C_M: \quad \begin{array}{ccc} M & \longrightarrow & M \\ m & \longmapsto & vm \end{array} \quad \text{so that} \quad C_{M \otimes N} = (\check{R}_{MN}\check{R}_{NM})^{-1}(C_M \otimes C_N) \quad (1.27)$$

(see [7, Prop. 3.2]). If M is a $U_h\mathfrak{g}$ -module generated by a highest weight vector v_λ^+ of weight λ , then

$$C_M = q^{-\langle \lambda, \lambda + 2\rho \rangle} \text{id}_M, \quad \text{where} \quad q = e^{h/2} \quad (1.28)$$

where E_{ij} is the matrix with 1 in the (i,j) entry and 0 elsewhere. A Cartan subalgebra of \mathfrak{gl}_r is

$$\mathfrak{h}_{\mathfrak{gl}} = \{x \in \mathfrak{gl}_r \mid x \text{ is diagonal}\} \quad \text{with basis} \quad \{E_{11}, E_{22}, \dots, E_{rr}\},$$

and the dual basis $\{\varepsilon_1, \dots, \varepsilon_r\}$ of $\mathfrak{h}_{\mathfrak{gl}}^*$ is specified by

$$\varepsilon_i: \mathfrak{h}_{\mathfrak{gl}} \rightarrow \mathbb{C} \quad \text{given by} \quad \varepsilon_i(E_{jj}) = \delta_{ij}.$$

The form

$$\langle \cdot, \cdot \rangle: \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{C} \quad \text{given by} \quad \langle x, y \rangle = \text{tr}_V(xy) \quad (2.1)$$

is a nondegenerate ad-invariant symmetric bilinear form on \mathfrak{g} such that the restriction to $\mathfrak{h}_{\mathfrak{gl}}$ is a nondegenerate form $\langle \cdot, \cdot \rangle: \mathfrak{h}_{\mathfrak{gl}} \otimes \mathfrak{h}_{\mathfrak{gl}} \rightarrow \mathbb{C}$. Since $\langle \cdot, \cdot \rangle$ is nondegenerate, the map $\nu: \mathfrak{h}_{\mathfrak{gl}} \rightarrow \mathfrak{h}_{\mathfrak{gl}}^*$ given by $\nu(h) = \langle h, \cdot \rangle$ is a vector space isomorphism which induces a nondegenerate form $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}_{\mathfrak{gl}}^*$. Further,

$$\{E_{11}, \dots, E_{rr}\} \quad \text{and} \quad \{\varepsilon_1, \dots, \varepsilon_r\} \quad \text{are orthonormal bases of } \mathfrak{h}_{\mathfrak{gl}} \text{ and } \mathfrak{h}_{\mathfrak{gl}}^*,$$

respectively. A Cartan subalgebra of \mathfrak{sl}_r is

$$\mathfrak{h}_{\mathfrak{sl}} = (E_{11} + \dots + E_{rr})^\perp = \{x \in \mathfrak{h}_{\mathfrak{gl}} \mid \langle x, E_{11} + \dots + E_{rr} \rangle = 0\},$$

the orthogonal subspace to $\mathbb{C}(E_{11} + \dots + E_{rr})$. The *dominant integral weights* for \mathfrak{gl}_r ,

$$P^+ = \{\lambda_1 \varepsilon_1 + \dots + \lambda_r \varepsilon_r \mid \lambda_i \in \mathbb{Z}, \lambda_1 \geq \dots \geq \lambda_r\},$$

index the irreducible finite-dimensional representations $L(\lambda)$ of \mathfrak{gl}_r . The irreducible finite-dimensional representations of \mathfrak{sl}_r are

$$L(\bar{\lambda}) = \text{Res}_{\mathfrak{sl}_r}^{\mathfrak{gl}_r}(L(\lambda)),$$

where $\bar{\lambda}$ is the orthogonal projection of λ to $\mathfrak{h}_{\mathfrak{sl}}^* = (\varepsilon_1 + \dots + \varepsilon_r)^\perp$.

The matrix units $\{E_{ij} \mid 1 \leq i, j \leq r\}$ form a basis of \mathfrak{gl}_r for which the dual basis with respect to the form in (2.1) is $\{E_{ji} \mid 1 \leq i, j \leq r\}$. So

$$\gamma^{\mathfrak{gl}} = \sum_{1 \leq i, j \leq r} E_{ij} \otimes E_{ji} = \sum_{\substack{1 \leq i, j \leq r \\ i \neq j}} E_{ij} \otimes E_{ji} + \sum_{i=1}^r E_{ii} \otimes E_{ii}, \quad \text{and} \quad (2.2)$$

$$\gamma^{\mathfrak{sl}} = \gamma^{\mathfrak{gl}} - E_+ \otimes E_+, \quad \text{where} \quad E_+ = E_{11} + \dots + E_{rr}. \quad (2.3)$$

If the Casimir for \mathfrak{gl}_r ,

$$\kappa^{\mathfrak{gl}} = \sum_{1 \leq i, j \leq r} E_{ij} E_{ji}, \quad \text{acts on} \quad L(\lambda) \quad \text{by the constant} \quad \kappa^{\mathfrak{gl}}(\lambda)$$

then the Casimir for \mathfrak{sl}_r ,

$$\kappa^{\mathfrak{sl}} = \kappa^{\mathfrak{gl}} - E_+ E_+, \quad \text{acts on} \quad L(\bar{\lambda}) \quad \text{by the constant} \quad \kappa^{\mathfrak{gl}}(\lambda) - \frac{1}{r} |\lambda|^2, \quad (2.4)$$

where $|\lambda| = \lambda_1 + \dots + \lambda_r$.

Let $V = \mathbb{C}^N$. The Lie algebras $\mathfrak{g} = \mathfrak{so}_N$ or \mathfrak{sp}_N are given by

$$\mathfrak{g} = \{x \in \mathfrak{gl}(V) \mid (xv_1, v_2) + (v_1, xv_2) = 0 \text{ for all } v_1, v_2 \in V\},$$

$$\begin{aligned} & \{F_{i,i}, F_{-i,i}, F_{i,-i} \mid 0 < i \in \hat{V}\} \cup \{F_{\pm i, \pm j} \mid 0 < i < j \in \hat{V}\} & \text{if } \mathfrak{g} = \mathfrak{sp}_{2r}, \\ & \{F_{i,i} \mid 0 < i \in \hat{V}\} \cup \{F_{\pm i, \pm j} \mid 0 < i < j \in \hat{V}\} & \text{if } \mathfrak{g} = \mathfrak{so}_{2r}. \end{aligned}$$

With respect to the nondegenerate ad-invariant symmetric bilinear form $\langle, \rangle: \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{C}$ given in (2.10), $\langle x, y \rangle = \frac{1}{2} \text{tr}_V(xy)$, the dual basis with respect to \langle, \rangle is

$$F_{ij}^* = F_{ji} \quad \text{if } i \neq -j, \quad \text{and} \quad F_{i,-i}^* = \frac{1}{2} F_{-i,i}.$$

The sets

$$\begin{aligned} & \{F_{-i,-i} \mid 0 < i \in \hat{V}\} \cup \{F_{\pm i, \pm j} \mid 0 < j < i \in \hat{V}\} \cup \{F_{\pm i, 0} \mid 0 < i \in \hat{V}\} & \text{if } \mathfrak{g} = \mathfrak{so}_{2r+1}, \\ & \{F_{-i,-i}, F_{-i,i}, F_{i,-i} \mid 0 < i \in \hat{V}\} \cup \{F_{\pm i, \pm j} \mid 0 < j < i \in \hat{V}\} & \text{if } \mathfrak{g} = \mathfrak{sp}_{2r}, \\ & \{F_{-i,-i} \mid 0 < i \in \hat{V}\} \cup \{F_{\pm i, \pm j} \mid 0 < j < i \in \hat{V}\} & \text{if } \mathfrak{g} = \mathfrak{so}_{2r}, \end{aligned}$$

are alternate bases, and $F_{i,-i} = 0$ when $\mathfrak{g} = \mathfrak{so}_{2r+1}$ or $\mathfrak{g} = \mathfrak{so}_{2r}$. So

$$2\gamma = \sum_{i,j \in \hat{V}} F_{ij} \otimes F_{ji}^* + \sum_{i \in \hat{V}} F_{i,-i} \otimes F_{i,-i}^* = \sum_{i,j \in \hat{V}} F_{ij} \otimes F_{ji}. \quad (2.11)$$

To compute the value $\frac{1}{2} \langle \lambda, \lambda + 2\rho \rangle$ in (1.18) choose positive roots

$$R^+ = \begin{cases} \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq r\} \cup \{\varepsilon_i \mid 1 \leq i \leq r\}, & \text{for } \mathfrak{g} = \mathfrak{so}_{2r+1}, \\ \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq r\} \cup \{2\varepsilon_i \mid 1 \leq i \leq r\}, & \text{for } \mathfrak{g} = \mathfrak{sp}_{2r}, \\ \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq r\}, & \text{for } \mathfrak{g} = \mathfrak{so}_{2r}, \end{cases} \quad (2.12)$$

Since

$$\begin{aligned} & \sum_{1 \leq i < j \leq r} (\varepsilon_i - \varepsilon_j) + \sum_{1 \leq i < j \leq r} (\varepsilon_i + \varepsilon_j) + \sum_{i=1}^r \varepsilon_i + \sum_{i=1}^r \varepsilon_i \\ & = \sum_{i=1}^r (r - 2i + 1)\varepsilon_i + \sum_{i=1}^r (r - 1)\varepsilon_i + \sum_{i=1}^r \varepsilon_i + \sum_{i=1}^r \varepsilon_i, \end{aligned}$$

it follows that

$$2\rho = \sum_{i=1}^r (y - 2i + 1)\varepsilon_i, \quad \text{where } y = \langle \varepsilon_1, \varepsilon_1 + 2\rho \rangle = \begin{cases} 2r, & \text{if } \mathfrak{g} = \mathfrak{so}_{2r+1}, \\ 2r + 1, & \text{if } \mathfrak{g} = \mathfrak{sp}_{2r}, \\ 2r - 1, & \text{if } \mathfrak{g} = \mathfrak{so}_{2r}, \end{cases} \quad (2.13)$$

is the value by which the Casimir κ acts on $L(\varepsilon_1)$. Set $q = e^{h/2}$. The quantum dimension of V is

$$\dim_q(V) = \text{tr}_V(e^{h\rho}) = \epsilon + [y], \quad \text{where } [y] = \frac{q^y - q^{-y}}{q - q^{-1}}, \quad (2.14)$$

since, with respect to a weight basis of V , the eigenvalues of the diagonal matrix $e^{h\rho}$ are $e^{\frac{1}{2}h(y-2i+1)} = q^{(y-2i+1)}$.

Identify a weight $\lambda = \lambda_1 \varepsilon_1 + \cdots + \lambda_r \varepsilon_r$ with the configuration of boxes with λ_i boxes in row i . If b is a box in position (i, j) of λ then the *content* of b is

$$c(b) = j - i = \text{the diagonal number of } b, \quad (2.15)$$

so that

0	1	2
-1	0	1
-2		

are the contents of the boxes of $\lambda = 3\varepsilon_1 + 3\varepsilon_2 + \varepsilon_3$.

If $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n$, then

$$\langle \lambda, \lambda + 2\rho \rangle - \langle \lambda - \varepsilon_i, \lambda - \varepsilon_i + 2\rho \rangle = 2\lambda_i + 2\rho_i - 1 = y + 2\lambda_i - 2i = y + 2c(\lambda/\lambda^-),$$

where λ/λ^- is the box at the end of row i in λ . By induction,

$$\langle \lambda, \lambda + 2\rho \rangle = y|\lambda| + 2 \sum_{b \in \lambda} c(b), \quad (2.16)$$

for $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_r\varepsilon_r$ with $\lambda_i \in \mathbb{Z}$.

Let $L(\lambda)$ be the irreducible highest weight \mathfrak{g} -module with highest weight λ , and let $V = L(\varepsilon_1)$. Then, for $\mathfrak{g} = \mathfrak{so}_{2r+1}$, \mathfrak{sp}_{2r} or \mathfrak{so}_{2r} ,

$$V \cong V^* \quad \text{and} \quad V \otimes V \cong L(0) \oplus L(2\varepsilon_1) \oplus L(\varepsilon_1 + \varepsilon_2). \quad (2.17)$$

For each component in the decomposition of $V \otimes V$ the values by which $\gamma = \sum_{b \in B} b \otimes b^*$ acts (see (1.18)) are

$$\begin{aligned} \langle 0, 0 + 2\rho \rangle - \langle \varepsilon_1, \varepsilon_1 + 2\rho \rangle - \langle \varepsilon_1, \varepsilon_1 + 2\rho \rangle &= 0 - y - y = -2y, \\ \langle 2\varepsilon_1, 2\varepsilon_1 + 2\rho \rangle - \langle \varepsilon_1, \varepsilon_1 + 2\rho \rangle - \langle \varepsilon_1, \varepsilon_1 + 2\rho \rangle &= 4 + 2(y-1) - y - y = 2, \\ \langle \varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_2 + 2\rho \rangle - \langle \varepsilon_1, \varepsilon_1 + 2\rho \rangle - \langle \varepsilon_1, \varepsilon_1 + 2\rho \rangle &= 2 + (y-1) + (y-3) - y - y = -2. \end{aligned} \quad (2.18)$$

The second symmetric and exterior powers of V are

$$S^2(V) = \begin{cases} L(2\varepsilon_1) \oplus L(0), & \text{if } \mathfrak{g} = \mathfrak{so}_{2r+1} \text{ or } \mathfrak{so}_{2r}, \\ L(2\varepsilon_1), & \text{if } \mathfrak{g} = \mathfrak{sp}_{2r}, \end{cases} \quad (2.19)$$

and

$$\Lambda^2(V) = \begin{cases} L(\varepsilon_1 + \varepsilon_2), & \text{if } \mathfrak{g} = \mathfrak{so}_{2r+1} \text{ or } \mathfrak{so}_{2r}, \\ L(\varepsilon_1 + \varepsilon_2) \oplus L(0), & \text{if } \mathfrak{g} = \mathfrak{sp}_{2r}. \end{cases} \quad (2.20)$$

For all dominant integral weights λ

$$L(\lambda) \otimes V = \begin{cases} L(\lambda) \oplus \left(\bigoplus_{\lambda^\pm} L(\lambda^\pm) \right), & \text{if } \mathfrak{g} = \mathfrak{so}_{2r+1} \text{ and } \lambda_r > 0, \\ \bigoplus_{\lambda^\pm} L(\lambda^\pm), & \text{if } \mathfrak{g} = \mathfrak{sp}_{2r}, \mathfrak{g} = \mathfrak{so}_{2r}, \text{ or} \\ & \text{if } \mathfrak{g} = \mathfrak{so}_{2r+1} \text{ and } \lambda_r = 0, \end{cases} \quad (2.21)$$

where the sum over λ^\pm denotes a sum over all dominant weights obtained by adding or removing a box from λ (by a routine check using the product formula for Weyl characters in [4, VIII §9 Prop. 2]). If $\mathfrak{g} = \mathfrak{so}_{2r}$ then addition and removal of a box should include the possibility of addition and removal of a box marked with a $-$ sign, and removal of a box from row r when $\lambda_r = \frac{1}{2}$ changes λ_r to $-\frac{1}{2}$.

Preliminaries on quantum trace. This paragraph provides a brief review of quantum traces and quantum dimensions (see also [5, 4.2.9]) in the form suitable to our needs for the proofs of the main theorems of this section. If \mathfrak{g} is a finite-dimensional complex Lie algebra with a symmetric nondegenerate ad-invariant bilinear form, and $U_h\mathfrak{g}$ is the Drinfel'd-Jimbo quantum group corresponding to \mathfrak{g} , then both

$$U = U\mathfrak{g} \text{ with } \mathcal{R} = 1 \otimes 1 \text{ and } v = 1 \quad \text{and} \quad U = U_h\mathfrak{g} \text{ with } v = e^{-h\rho_u} \quad (2.22)$$

are ribbon Hopf algebras ([18, Corollary (2.15)]). For $U = U\mathfrak{g}$ or $U_h\mathfrak{g}$, let V be a finite-dimensional U -module, and let V^* be the dual module. Define

$$\begin{aligned} \text{ev}: V^* \otimes V &\longrightarrow \mathbf{1} & \text{and} & & \text{coev}: \mathbf{1} &\longrightarrow V \otimes V^* \\ \phi \otimes v &\longmapsto \phi(v) & & & 1 &\longmapsto \sum_i v_i \otimes v^i \end{aligned}$$

where $\{v_1, \dots, v_n\}$ is a basis of V and $\{v^1, \dots, v^n\}$ is the dual basis in V^* . Let E_V be the composition

$$E_V: V \otimes V^* \xrightarrow{v^{-1} \otimes 1} V \otimes V^* \xrightarrow{\check{R}_{VV^*}} V^* \otimes V \xrightarrow{\text{ev}} \mathbf{1} \xrightarrow{\text{coev}} V \otimes V^*,$$

so that E_V is a U -module homomorphism with image a submodule of $V \otimes V^*$ isomorphic to the trivial representation of U .

Let M be a U -module and let $\psi \in \text{End}_U(M \otimes V)$. Then, as operators on $M \otimes V \otimes V^*$,

$$(\text{id} \otimes E_V)(\psi \otimes \text{id})(\text{id} \otimes E_V) = (\text{id} \otimes \text{qtr}_V)(\psi) \otimes E_V, \quad (2.23)$$

where the *quantum trace* $(\text{id} \otimes \text{qtr}_V)(\psi): M \rightarrow M$ is given by

$$(\text{id} \otimes \text{qtr}_V)(\psi) = (\text{id} \otimes \text{tr}_V)((1 \otimes uv^{-1})\psi). \quad (2.24)$$

The special case when $M = \mathbf{1}$ and $\psi = \text{id}_V$ is the *quantum dimension* of V ,

$$\dim_q(V) = \text{qtr}_V(\text{id}_V), \quad \text{so that} \quad E_V^2 = \dim_q(V)E_V. \quad (2.25)$$

If $C_V: V \rightarrow V$ is the map defined in (1.27), then

$$(\text{id} \otimes \text{qtr}_V)(\check{R}_{VV}) = C_V^{-1} \quad (2.26)$$

(see, for example, [18, Prop. 3.11]). In the case $U = U\mathfrak{g}$, the ribbon element $v = 1$, so that $\text{qtr}_V(\varphi) = \text{tr}_V(\varphi)$ and $C_V = \text{id}_V$.

Remark 2.1. The identity (2.23) and the second identity in (2.25) are the source of the connection between quantum traces, the Jones basic construction and conditional expectations (see [11, Def. 2.6.6]). These tools are extremely powerful for the study of Temperley-Lieb algebras, Brauer algebras, BMW algebras, and other algebras which arise as tangle algebras (tensor power centralizer algebras).

2.1 The degenerate affine BMW algebra action

Define e_i in the degenerate affine braid algebra \mathcal{B}_k by

$$t_{s_i} y_i = y_{i+1} t_{s_i} - (1 - e_i), \quad \text{for } i = 1, 2, \dots, k-1, \quad (2.27)$$

so that, with $\gamma_{i,i+1}$ as in (1.7),

$$\gamma_{i,i+1}t_{s_i} = 1 - e_i. \quad (2.28)$$

By definition, the algebra \mathcal{B}_k is an algebra over a commutative base ring C . Fix constants

$$\epsilon = \pm 1 \quad \text{and} \quad z_0^{(\ell)} \in C \quad \text{for } \ell \in \mathbb{Z}_{\geq 0}.$$

The *degenerate affine Birman-Wenzl-Murakami (BMW) algebra* \mathcal{W}_k (with parameters ϵ and $z_0^{(\ell)}$) is the quotient of the degenerate affine braid algebra \mathcal{B}_k by the relations

$$e_i t_{s_i} = t_{s_i} e_i = \epsilon e_i, \quad e_i t_{s_{i-1}} e_i = e_i t_{s_{i+1}} e_i = \epsilon e_i, \quad (2.29)$$

$$e_1 y_1^\ell e_1 = z_0^{(\ell)} e_1, \quad e_i (y_i + y_{i+1}) = 0 = (y_i + y_{i+1}) e_i. \quad (2.30)$$

This algebra was introduced by Nazarov [21, §4] as the degenerate affine Wenzl algebra. The *degenerate affine Hecke algebra* \mathcal{H}_k is the quotient of \mathcal{W}_k by the relations

$$e_i = 0, \quad \text{for } i = 1, \dots, k-1. \quad (2.31)$$

Theorem 2.2. *Let $\Phi : \mathcal{B}_k \rightarrow \text{End}_{\mathfrak{g}}(M \otimes V^{\otimes k})$ be the representation defined in Theorem 1.2. In particular, \mathcal{B}_k is the degenerate affine braid algebra over the ring $C = Z(U_{\mathfrak{g}})$.*

(a) *Let \mathfrak{g} be \mathfrak{so}_{2r+1} , \mathfrak{sp}_{2r} or \mathfrak{so}_{2r} and $\gamma = \sum_b b \otimes b^*$ as in (2.11). Use notations for irreducible representations as in (2.21) and let y and ϵ be as in (2.13) and (2.5). Let*

$$z_0^{(\ell)} = \epsilon(\text{id} \otimes \text{tr}_V)((\frac{1}{2}y + \gamma)^\ell), \quad \text{for } \ell \in \mathbb{Z}_{\geq 0}.$$

Then $\Phi : \mathcal{B}_k \rightarrow \text{End}_U(M \otimes V^{\otimes k})$ factors to a representation of the degenerate affine BMW algebra \mathcal{W}_k .

(b) *If $\mathfrak{g} = \mathfrak{gl}_r$, $\gamma = \sum_b b \otimes b^*$ is as in (2.2), and $V = L(\varepsilon_1)$ then $\Phi : \mathcal{B}_k \rightarrow \text{End}_{U_{\mathfrak{g}}}(M \otimes V^{\otimes k})$ factors to a representation of the degenerate affine Hecke algebra. If $\mathfrak{g} = \mathfrak{sl}_r$, $\gamma = \sum_b b \otimes b^*$ is as in (2.3), and $V = L(\bar{\varepsilon}_1)$ then $\Phi' : \mathcal{B}_k \rightarrow \text{End}_{U_{\mathfrak{g}}}(M \otimes V^{\otimes k})$ given by*

$$\Phi'(t_{s_i}) = \Phi(t_{s_i}), \quad \Phi'(\gamma_{\ell,m}) = \frac{1}{r} + \Phi(\gamma_{\ell,m}), \quad \text{and} \quad \Phi'(\kappa_i) = \Phi(\kappa_i)$$

factors to a representation of the degenerate affine Hecke algebra.

Proof. (a) It is a consequence of [7, Prop. 1.2] that the $z_0^{(\ell)}$ are central elements of the enveloping algebra $U_{\mathfrak{g}}$.

The action of γ on the tensor product of two simple modules is given in (1.18), so the computations in (2.18) determine the action of γ on the components of $V \otimes V$. The decompositions of the second symmetric and exterior powers in (2.19) and (2.20) determine the action of t_{s_1} on $V \otimes V$. The operator $\Phi(e_1)$ is determined from $\Phi(t_{s_1})$ and $\Phi(\gamma)$ via (2.28),

$$\Phi(\gamma)\Phi(t_{s_1}) = 1 - \Phi(e_1).$$

In summary, $\Phi(t_{s_1})$, $\Phi(e_1)$, and $\Phi(\gamma)$ act on the components of $V \otimes V$ by

	$L(0)$	$L(2\varepsilon_1)$	$L(\varepsilon_1 + \varepsilon_2)$
$\Phi(\gamma_{1,2})$	$-y$	1	-1
$\Phi(t_{s_1})$	ϵ	1	-1
$\Phi(e_1)$	$1 + \epsilon y$	0	0

where y and γ are as in (2.13) and (1.16), respectively. The first relation in (2.29) follows.

Since $\dim(V) = \epsilon + y$, the first identity in (2.25) gives that

$$\Phi(e_1) = \epsilon E_V. \quad (2.32)$$

By (2.22), (2.23), and (2.26),

$$\begin{aligned} \Phi(e_i t_{s_{i-1}} e_i) &= \epsilon(1 \otimes E_V)(\check{R}_{VV} \otimes 1)(1 \otimes E_V)\epsilon = (\text{id} \otimes \text{tr}_V)(\check{R}_{VV}) \otimes E_V \\ &= C_V^{-1} \otimes E_V = \text{id} \otimes E_V = \epsilon \Phi(e_i), \end{aligned}$$

which establishes the second relation in (2.29). Since $y = \langle \epsilon_1, \epsilon_1 + 2\rho \rangle = \Phi(\kappa_i)$, it follows from (1.12) that $\Phi(y_1) = \Phi(\frac{1}{2}\kappa_1 + \gamma_{0,1}) = \frac{1}{2}y + \gamma$, and by (2.23),

$$\begin{aligned} \Phi(e_1 y_1^\ell e_1) &= \epsilon(\text{id} \otimes E_V)(\frac{1}{2}y + \gamma)^\ell \epsilon(\text{id} \otimes E_V) = (\text{id} \otimes \text{tr}_V)((\frac{1}{2}y + \gamma)^\ell) \otimes E_V \\ &= \epsilon(\text{id} \otimes \text{tr}_V)((\frac{1}{2}y + \gamma)^\ell) \Phi(e_1) = z_0^{(\ell)} \Phi(e_1), \end{aligned} \quad (2.33)$$

which gives the first relation in (2.30). Since the y_i commute and $t_{s_i}(y_i + y_{i+1}) = (y_i + y_{i+1})t_{s_i}$, it follows that

$$e_i(y_i + y_{i+1}) = (t_{s_i} y_i - y_{i+1} t_{s_i} + 1)(y_i + y_{i+1}) = (y_i + y_{i+1})(t_{s_i} y_i - y_{i+1} t_{s_i} + 1) = (y_i + y_{i+1})e_i.$$

For $b \in U\mathfrak{g}$ or $U\mathfrak{g} \otimes U\mathfrak{g}$, let b_i and $b_{i,i+1}$ denote the action of an element b on the i th, respectively i th and $(i+1)$ st, factors of V in $M \otimes V^{\otimes(i+1)}$. Then, as operators on $M \otimes V^{\otimes(i+1)}$,

$$\begin{aligned} (y_i + y_{i+1})e_i &= \left(\frac{1}{2}\kappa_i + \sum_{r=0}^{i-1} \gamma_{r,i} + \frac{1}{2}\kappa_{i+1} + \sum_{r=0}^i \gamma_{r,i+1} \right) e_i = \left(\frac{1}{2}\Delta(\kappa)_{i,i+1} + \sum_{r=0}^{i-1} (\gamma_{r,i} + \gamma_{r,i+1}) \right) e_i \\ &= \left(\frac{1}{2}\Delta(\kappa)_{i,i+1} + \sum_{r=0}^{i-1} \sum_b b \otimes \Delta(b^*)_{i,i+1} \right) e_i = 0, \end{aligned}$$

because e_i is a projection onto $L(0)$ and the action of b^* and κ on $L(0)$ is 0.

(b) In the case where $\mathfrak{g} = \mathfrak{gl}_r$ and $V = L(\epsilon_1)$,

$$V \otimes V = L(2\epsilon_1) \oplus L(\epsilon_1 + \epsilon_2), \quad \text{with } \Lambda^2(V) = L(\epsilon_1 + \epsilon_2) \quad \text{and} \quad S^2(V) = L(2\epsilon_1).$$

So by (1.18),

	$L(2\epsilon_1)$	$L(\epsilon_1 + \epsilon_2)$		$\Phi(e_1) = \Phi(\gamma) - \Phi(t_{s_1}) = 0.$	(2.34)
$\Phi(\gamma_{1,2})$	1	-1	and		
$\Phi(t_{s_1})$	1	-1			

In the case where $\mathfrak{g} = \mathfrak{sl}_r$ and $V = L(\bar{\epsilon}_1)$,

$$V \otimes V = L(\overline{2\epsilon_1}) \oplus L(\overline{\epsilon_1 + \epsilon_2}), \quad \text{with } \Lambda^2(V) = L(\overline{\epsilon_1 + \epsilon_2}) \quad \text{and} \quad S^2(V) = L(\overline{2\epsilon_1}).$$

As the map $\phi : \mathcal{B}_k \rightarrow \mathcal{B}_k$ given by

$$t_{s_i} \mapsto t_{s_i}, \quad \gamma_{i,j} \mapsto \gamma_{i,j} - a, \quad \kappa_i \mapsto \kappa_i, \quad \text{for fixed } a \in C$$

is an automorphism, the result follows from (2.34) and (2.4). □

Remark 2.3. Fix $b_1, \dots, b_r \in C$. The *degenerate cyclotomic BMW algebra* $\mathcal{W}_{r,k}(b_1, \dots, b_r)$ is the degenerate affine BMW algebra with the additional relation

$$(y_1 - b_1) \cdots (y_1 - b_r) = 0. \quad (2.35)$$

The *degenerate cyclotomic Hecke algebra* $\mathcal{H}_{r,k}(b_1, \dots, b_r)$ is the degenerate affine Hecke algebra \mathcal{H}_k with the additional relation (2.35). In Theorem 2.2, if $\Phi(y_1)$ has eigenvalues u_1, \dots, u_r then Φ is a representation of $\mathcal{W}_{r,k}(u_1, \dots, u_r)$ or $\mathcal{H}_{r,k}(u_1, \dots, u_r)$.

Remark 2.4. In general, for any constants a_0, a , and c , the map $\phi : \mathcal{B}_k \rightarrow \mathcal{B}_k$ given by

$$t_{s_i} \mapsto t_{s_i}, \quad \gamma_{i,j} \mapsto \gamma_{i,j} - c, \quad \kappa_0 \mapsto \kappa_0 - a_0, \quad \text{and} \quad \kappa_j \mapsto \kappa_j - a, \quad \text{for } j = 1, \dots, k,$$

is an automorphism. So, following the proof of Theorem 2.2(b), $\Phi' : \mathcal{B}_k \rightarrow \text{End}_{U_{\mathfrak{g}}}(M \otimes V^{\otimes k})$ given by

$$\begin{aligned} \Phi'(t_{s_i}) &= \Phi(t_{s_i}), & \Phi'(\gamma_{\ell,m}) &= \frac{1}{r} + \Phi(\gamma_{\ell,m}), \\ \Phi'(\kappa_0) &= a_0 + \Phi(\kappa_0), & \text{and} \quad \Phi'(\kappa_j) &= a + \Phi(\kappa_j) \quad \text{for } j = 1, \dots, k, \end{aligned}$$

also extends to a representation of \mathcal{H}_k when $\mathfrak{g} = \mathfrak{sl}_r$. When $M = L(\mu)$ is a finite-dimensional highest weight module taking $a_0 = \frac{|\mu|}{r}$ and $a = \frac{1}{r}$ is combinatorially convenient.

2.2 The affine BMW algebra action

Let C be a commutative ring and let CB_k be the group algebra of the affine braid group. Fix constants

$$q, z \in C \quad \text{and} \quad Z_0^{(\ell)} \in C, \quad \text{for } \ell \in \mathbb{Z},$$

with q and z invertible. Let $Y_i = zX^{\varepsilon_i}$ so that

$$Y_1 = zX^{\varepsilon_1}, \quad Y_i = T_{i-1}Y_{i-1}T_{i-1}, \quad \text{and} \quad Y_iY_j = Y_jY_i, \quad \text{for } 1 \leq i, j \leq k. \quad (2.36)$$

In the affine braid group

$$T_iY_iY_{i+1} = Y_iY_{i+1}T_i. \quad (2.37)$$

Assume $q - q^{-1}$ is invertible in C . Define $E_i \in CB_k$ by

$$T_iY_i = Y_{i+1}T_i - (q - q^{-1})Y_{i+1}(1 - E_i). \quad (2.38)$$

The *affine BMW algebra* W_k is the quotient of the group algebra CB_k by the relations

$$E_iT_i^{\pm 1} = T_i^{\pm 1}E_i = z^{\mp 1}E_i, \quad E_iT_{i-1}^{\pm 1}E_i = E_iT_{i+1}^{\pm 1}E_i = z^{\pm 1}E_i, \quad (2.39)$$

$$E_1Y_1^\ell E_1 = Z_0^{(\ell)}E_1, \quad E_iY_iY_{i+1} = E_i = Y_iY_{i+1}E_i. \quad (2.40)$$

For all practical purposes this algebra was defined in [15], see also [22, §6.2]. Left multiplying (2.38) by Y_{i+1}^{-1} and using the second identity in (2.36) shows that (2.38) is equivalent to $T_i - T_i^{-1} = (q - q^{-1})(1 - E_i)$. So

$$E_i = 1 - \frac{T_i - T_i^{-1}}{q - q^{-1}}, \quad \text{and} \quad E_i^2 = \left(1 + \frac{z - z^{-1}}{q - q^{-1}}\right) E_i \quad (2.41)$$

follows by multiplying the first equation in (2.41) by E_i and using (2.39).

The *affine Hecke algebra* H_k is the affine BMW algebra W_k with the additional relations

$$E_i = 0, \quad \text{for } i = 1, \dots, k-1. \quad (2.42)$$

Theorem 2.5. Let $\Phi : CB_k \rightarrow \text{End}_{U_h\mathfrak{g}}(M \otimes V^{\otimes k})$ be the representation defined in Theorem 1.3. In particular, CB_k is the group algebra of B_k over the ring $C = Z(U_h\mathfrak{g})$.

(a) Let \mathfrak{g} be \mathfrak{so}_{2r+1} , \mathfrak{sp}_{2r} or \mathfrak{so}_{2r} and let $\gamma = \sum_b b \otimes b^*$ is as in (2.11). Let y and ϵ be as in (2.13) and (2.5), $z = \epsilon q^y$, and

$$Z_0^{(\ell)} = \epsilon (\text{id} \otimes \text{qtr}_V)((z\mathcal{R}_{21}\mathcal{R})^\ell), \quad \text{for } \ell \in \mathbb{Z}.$$

Then $\Phi : CB_k \rightarrow \text{End}_U(M \otimes V^{\otimes k})$ factors to a representation of the affine BMW algebra W_k .

(b) If $\mathfrak{g} = \mathfrak{gl}_r$, $\gamma = \sum_b b \otimes b^*$ is as in (2.2), and $V = L(\varepsilon_1)$, then $\Phi : CB_k \rightarrow \text{End}_U(M \otimes V^{\otimes k})$ factors to a representation of the affine Hecke algebra. If $\mathfrak{g} = \mathfrak{sl}_r$, $\gamma = \sum_b b \otimes b^*$ is as in (2.3), and $V = L(\bar{\varepsilon}_1)$, then

$$\Phi' : CB_k \rightarrow \text{End}_U(M \otimes V^{\otimes k}) \quad \text{given by} \quad \Phi'(T_i) = q^{1/r} \Phi(T_i) \quad \text{and} \quad \Phi'(X^{\varepsilon_i}) = \Phi(X^{\varepsilon_i}),$$

factors to a representation of the affine Hecke algebra.

Proof. (a) It is a consequence of [7, Prop. 1.2] that the $Z_0^{(\ell)}$ are central elements of the quantum group $U_h\mathfrak{g}$.

By (1.29), the computations in (2.18) determine the action of \check{R}_{VV}^2 on the components of $V \otimes V$. The operator $\Phi(T_1) = \check{R}_{VV}$ is the square root of \check{R}_{VV}^2 and, at $q = 1$, specializes to t_{s_1} , the operator that switches the factors in $V \otimes V$. Thus equations (2.19) and (2.20) determine the sign of $\Phi(T_1)$ on each component. The operator $\Phi(E_1)$ is determined from $\Phi(T_1)$ via the first identity in (2.41),

$$\Phi(E_i) = 1 - \frac{\Phi(T_i) - \Phi(T_i^{-1})}{q - q^{-1}}.$$

Then \check{R}_{VV}^2 , $\Phi(T_1)$ and $\Phi(E_1)$ act on the components of $V \otimes V$ by

	$L(0)$	$L(2\varepsilon_1)$	$L(\varepsilon_1 + \varepsilon_2)$
\check{R}_{VV}^2	q^{-2y}	q^2	q^{-2}
$\Phi(T_1)$	ϵq^{-y}	q	$-q^{-1}$
$\Phi(E_1)$	$1 + \epsilon[y]$	0	0

where $[y] = \frac{q^y - q^{-y}}{q - q^{-1}}$.

The first relation in (2.39) follows from

$$\Phi(E_1 T_1) = \epsilon q^{-y} \Phi(E_1) = z^{-1} \Phi(E_1).$$

Since $\dim_q(V) = \epsilon + [y]$, (2.25) gives

$$\Phi(E_1) = \epsilon E_V. \tag{2.43}$$

By (2.23), (2.26), (1.28), and (2.13),

$$\begin{aligned} \Phi(E_i T_{i-1} E_i) &= \epsilon (1 \otimes E_V) (\check{R}_{VV} \otimes 1) (1 \otimes E_V) \epsilon = (\text{id} \otimes \text{qtr}_V) (\check{R}_{VV}) \otimes E_V \\ &= C_V^{-1} \otimes E_V = q^{(\varepsilon_1, \varepsilon_1 + 2\rho)} (\text{id} \otimes E_V) = q^y \epsilon \Phi(E_i) = z \Phi(E_i). \end{aligned}$$

This establishes the second relation in (2.39). By (2.23),

$$\begin{aligned} \Phi(E_1 Y_1^\ell E_1) &= \epsilon (\text{id} \otimes E_V) (z\mathcal{R}_{21}\mathcal{R})^\ell \epsilon (\text{id} \otimes E_V) = (\text{id} \otimes \text{qtr}_V) ((z\mathcal{R}_{21}\mathcal{R})^\ell) \otimes E_V \\ &= \epsilon (\text{id} \otimes \text{qtr}_V) ((z\mathcal{R}_{21}\mathcal{R})^\ell) \Phi(E_1) = Z_0^{(\ell)} \Phi(E_1), \end{aligned} \tag{2.44}$$

which gives the first relation in (2.40). Since the Y_i commute and $T_i Y_i Y_{i+1} = Y_i Y_{i+1} T_i$,

$$E_i Y_i Y_{i+1} = \left(1 - \frac{T_i - T_i^{-1}}{q - q^{-1}}\right) Y_i Y_{i+1} = Y_i Y_{i+1} \left(1 - \frac{T_i - T_i^{-1}}{q - q^{-1}}\right) = Y_i Y_{i+1} E_i.$$

The proof that $E_i Y_i Y_{i+1} = E_i$ is exactly as in the proof of [22, Thm. 6.1(c)]: Since $\Phi(E_1) = \epsilon E_V$, using $E_1 T_1 = z^{-1} E_1$ and the pictorial equalities

it follows that $\Phi(E_1 Y_1 Y_2 T_1^{-1}) = \epsilon(1 \otimes E_V) \Phi(z X^{\epsilon_1}) \Phi(z T_1 X^{\epsilon_1})$ acts as

$$\epsilon z^2 z^{-1} \cdot \check{R}_{L(0), M} \check{R}_{M, L(0)} (\text{id}_M \otimes E_V).$$

By (1.27), this is equal to

$$\epsilon z (C_M \otimes C_{L(0)}) C_{M \otimes L(0)}^{-1} (\text{id}_M \otimes E_V) = \epsilon z \cdot C_M C_M^{-1} (\text{id}_M \otimes E_V) = z \cdot \Phi(E_1) = \Phi(E_1 T_1^{-1}),$$

so that $\Phi(E_1 Y_1 Y_2 T_1^{-1}) = \Phi(E_1 T_1^{-1})$. This establishes the second relation in (2.40).

(b) In the case where $\mathfrak{g} = \mathfrak{gl}_r$ and $V = L(\varepsilon_1)$,

$$V \otimes V = L(2\varepsilon_1) \oplus L(\varepsilon_1 + \varepsilon_2) \quad \text{with } S^2(V) = L(2\varepsilon_1) \quad \text{and} \quad \Lambda^2(V) = L(\varepsilon_1 + \varepsilon_2).$$

So by (1.29),

	$L(2\varepsilon_1)$	$L(\varepsilon_1 + \varepsilon_2)$
$\Phi(\check{R}_{VV}^2)$	q^2	q^{-2}
$\Phi(T_1)$	q	$-q^{-1}$

so that $\Phi(E_1) = 1 - \frac{\Phi(T_1) - \Phi(T_1)^{-1}}{q - q^{-1}} = 0.$ (2.45)

In the case where $\mathfrak{g} = \mathfrak{sl}_r$ and $V = L(\bar{\varepsilon}_1)$,

$$V \otimes V = L(2\bar{\varepsilon}_1) \oplus L(\bar{\varepsilon}_1 + \bar{\varepsilon}_2), \quad \text{with } \Lambda^2(V) = L(\bar{\varepsilon}_1 + \bar{\varepsilon}_2) \quad \text{and} \quad S^2(V) = L(2\bar{\varepsilon}_1).$$

Since the map $\phi : B_k \rightarrow B_k$ given by

$$T_i \mapsto a T_i, \quad X^{\varepsilon_i} \mapsto X^{\varepsilon_i}, \quad \text{for invertible } a \in C$$

is an automorphism, the result then follows from (2.45) and (2.4) (also see [18, Prop. 4.4]). \square

Remark 2.6. Fix $b_1, \dots, b_r \in C$. The *cyclotomic BMW algebra* $W_{r,k}(b_1, \dots, b_r)$ is the affine BMW algebra W_k with the additional relation

$$(Y_1 - b_1) \cdots (Y_1 - b_r) = 0. \quad (2.46)$$

The *cyclotomic Hecke algebra* $H_{r,k}(b_1, \dots, b_r)$ is the affine Hecke algebra H_k with the additional relation (2.46). In Theorem 2.5, if $\Phi(Y_1)$ has eigenvalues u_1, \dots, u_r , then Φ is a representation of $W_{r,k}(u_1, \dots, u_r)$ or $H_{r,k}(u_1, \dots, u_r)$.

3 Central element transfer via Schur-Weyl duality

In Theorem 2.2 and Theorem 2.5, the parameters

$$z_0^{(\ell)} = \epsilon(\text{id} \otimes \text{tr}_V)((\frac{1}{2}y + \gamma)^\ell) \quad \text{and} \quad Z_0^{(\ell)} = \epsilon(\text{id} \otimes \text{qtr}_V)((z\mathcal{R}_{21}\mathcal{R})^\ell)$$

of the degenerate affine BMW algebra and affine BMW algebra, respectively, arise naturally from the action on tensor space. These are central elements of the enveloping algebra $U\mathfrak{g}$ and the quantum group $U_h\mathfrak{g}$, respectively:

$$z_0^{(\ell)} \in Z(U\mathfrak{g}) \quad \text{and} \quad Z_0^{(\ell)} \in Z(U_h\mathfrak{g}).$$

The Harish-Chandra isomorphism provides isomorphisms between the centers $Z(U\mathfrak{g})$ or $Z(U_h\mathfrak{g})$ and rings of symmetric functions. In this section we show how to use the recursive formulas of [21] and [3] for the central elements $z_k^{(\ell)}$ and $Z_k^{(\ell)}$ in the degenerate affine and affine BMW algebras (formulas (3.4) and (3.13)) to determine the Harish-Chandra images of $z_0^{(\ell)}$ and $Z_0^{(\ell)}$.

Preliminaries on the Harish-Chandra isomorphisms. Let \mathfrak{g} be a finite-dimensional complex Lie algebra with a symmetric nondegenerate ad-invariant bilinear form. The triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ (see [4, VII §8 no. 3 Prop. 9]) yields triangular decompositions of both the enveloping algebra $U = U\mathfrak{g}$ and the quantum group $U = U_h\mathfrak{g}$ in the form $U = U^-U_0U^+$. If $U = U_h\mathfrak{g}$ then $U_0 = \text{span}\{K^{\lambda^\vee} \mid \lambda^\vee \in \mathfrak{h}_\mathbb{Z}\}$ with $K^{\lambda^\vee}K^{\nu^\vee} = K^{\lambda^\vee + \nu^\vee}$, where $\mathfrak{h}_\mathbb{Z}$ is a lattice in \mathfrak{h} . Alternatively,

$$U_0 = U\mathfrak{h} = \mathbb{C}[h_1, \dots, h_r] \quad \text{if } U = U\mathfrak{g} \quad \text{and} \quad U_0 = \mathbb{C}[L_1^{\pm 1}, \dots, L_r^{\pm 1}] \quad \text{if } U = U_h\mathfrak{g},$$

where h_1, \dots, h_r is a basis of $\mathfrak{h}_\mathbb{Z}$, and $L_i = K^{h_i} = q^{h_i}$. The Weyl group W_0 of \mathfrak{g} acts on $U\mathfrak{h}$ since it acts $\mathfrak{h}_\mathbb{Z}$, and acts on U_0 for $U_h\mathfrak{g}$ by $wK^{\lambda^\vee} = K^{w\lambda^\vee}$, for $w \in W_0$ and $\lambda^\vee \in \mathfrak{h}_\mathbb{Z}$.

For $\mu \in \mathfrak{h}^*$, define the ring homomorphisms $\text{ev}_\mu : U_0 \rightarrow \mathbb{C}$ by

$$\text{ev}_\mu(h) = \langle \mu, h \rangle \quad \text{and} \quad \text{ev}_\mu(K^{\lambda^\vee}) = q^{\langle \mu, \lambda^\vee \rangle} \quad (3.1)$$

for $h \in \mathfrak{h}$ and K^{λ^\vee} with $\lambda^\vee \in \mathfrak{h}_\mathbb{Z}$. For $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$ as in (1.17), let σ_ρ be the algebra automorphism given by

$$\sigma_\rho(h_i) = h_i + \langle \rho, h_i \rangle \quad \text{and} \quad \sigma_\rho(L_i) = q^{\langle \rho, h_i \rangle} L_i. \quad (3.2)$$

Define a vector space homomorphism by

$$\pi_0 : U \longrightarrow U_0 \quad \text{by} \quad \pi_0 = \varepsilon^- \otimes \text{id} \otimes \varepsilon^+ : U_- \otimes U_0 \otimes U_+ \longrightarrow U_0, \quad (3.3)$$

where $\varepsilon^- : U_- \rightarrow \mathbb{C}$ and $\varepsilon^+ : U_+ \rightarrow \mathbb{C}$ are the algebra homomorphisms determined by

$$\begin{aligned} \varepsilon^-(y) = 0 \quad \text{and} \quad \varepsilon^+(x) = 0, \quad \text{for } x \in \mathfrak{n}^+ \text{ and } y \in \mathfrak{n}^-, \text{ or} \\ \varepsilon^-(F_i) = 0 \quad \text{and} \quad \varepsilon^+(E_i) = 0, \quad \text{for } i = 1, \dots, n. \end{aligned}$$

The following important theorem says that both the center of $U\mathfrak{g}$ and the center of $U_h\mathfrak{g}$ are isomorphic to rings of symmetric functions.

Theorem 3.1 (Harish-Chandra/Chevalley isomorphism, [4, VII §8 no. 5 Thm. 2] and [5, Thm. 9.1.6]). *Let $U = U\mathfrak{g}$ or $U_{\mathfrak{h}}\mathfrak{g}$, so that*

$$U_0 = \mathbb{C}[h_1, \dots, h_r] \quad \text{if } U = U\mathfrak{g} \quad \text{and} \quad U_0 = \mathbb{C}[L_1^{\pm 1}, \dots, L_r^{\pm 1}] \quad \text{if } U = U_{\mathfrak{h}}\mathfrak{g}.$$

Let $L(\mu)$ denote the irreducible U -module of highest weight μ . Then the restriction of π_0 to the center of U ,

$$\begin{array}{ccc} \pi_0: Z(U) & \longrightarrow & \sigma_\rho(U_0^{W_0}), \\ z & \longmapsto & \sigma_\rho(s) \end{array} \quad \text{is an algebra isomorphism,}$$

where $s \in U_0^{W_0}$ is the symmetric function determined by

$$z \quad \text{acts on } L(\mu) \text{ by} \quad \text{ev}_\mu(\sigma_\rho(s)) = \text{ev}_{\mu+\rho}(s), \quad \text{for } \mu \in \mathfrak{h}^*.$$

3.1 Central elements $z_V^{(\ell)}$

Let $z_0^{(\ell)}$ and ϵ be the parameters of the degenerate affine BMW algebra \mathcal{W}_k . Let u be a variable and define $z_i^{(\ell)} \in \mathcal{W}_k$ for $i = 1, \dots, k-1$ by

$$z_i(u) + \epsilon u - \frac{1}{2} = (z_0(u) + \epsilon u - \frac{1}{2}) \prod_{j=1}^i \frac{(u + y_j - 1)(u + y_j + 1)(u - y_j)^2}{(u + y_j)^2(u - y_j + 1)(u - y_j - 1)}, \quad (3.4)$$

where

$$z_i(u) = \sum_{\ell \in \mathbb{Z}_{\geq 0}} z_i^{(\ell)} u^{-\ell}, \quad \text{for } i = 0, 1, \dots, k-1.$$

The following proposition from [21, Lemma 3.8] is proved also in [6, Theorem 3.2 and Remark 3.4]

Proposition 3.2. *In the degenerate affine BMW algebra \mathcal{W}_k ,*

$$e_{i+1} y_{i+1}^\ell e_{i+1} = z_i^{(\ell)} e_{i+1}, \quad \text{for } i = 0, \dots, k-1 \text{ and } \ell \in \mathbb{Z}_{\geq 0}.$$

The following theorem uses the identity (3.4) and the action of the degenerate affine BMW algebra on tensor space to provide a formula for the Harish-Chandra images of the central elements $z_V^{(\ell)} = \epsilon (\text{id} \otimes \text{tr}_V) \left(\left(\frac{1}{2}y + \gamma \right)^\ell \right)$ in the enveloping algebra $U\mathfrak{g}$ for orthogonal and symplectic Lie algebras \mathfrak{g} . By Theorem 2.2 these particular central elements are natural parameters for the degenerate affine BMW algebras. The concept of the proof of Theorem 3.3 is, at its core, the same as the pattern taken by Nazarov for the proof of [21, Theorem 3.9].

Theorem 3.3. *Let $\mathfrak{g} = \mathfrak{so}_{2r+1}$, \mathfrak{sp}_{2r} or \mathfrak{so}_{2r} , use notations for \mathfrak{h}^* as in (2.5)-(2.16) and let h_1, \dots, h_r be the basis of \mathfrak{h} dual to the orthonormal basis $\varepsilon_1, \dots, \varepsilon_r$ of \mathfrak{h}^* (so that $h_i = F_{ii}$, where F_{ii} is as in (2.8)). With respect to the form \langle, \rangle in (2.10), let $\gamma = \sum_b b \otimes b^*$ as in (1.16). Let y and ϵ be as in (2.13) and (2.5), and let $V = L(\varepsilon_1)$. Let $z_V^{(\ell)}$ be the central elements in $U\mathfrak{g}$ defined by*

$$z_V^{(\ell)} = \epsilon (\text{id} \otimes \text{tr}_V) \left(\left(\frac{1}{2}y + \gamma \right)^\ell \right), \quad \text{and write} \quad z_V(u) = \sum_{i \in \mathbb{Z}_{\geq 0}} z_V^{(i)} u^{-i}.$$

Then

$$\pi_0(z_V(u) + \epsilon u - \frac{1}{2}) = (\epsilon u + \frac{1}{2}) \frac{(u + \frac{1}{2}y - r)}{(u - \frac{1}{2}y + r)} \sigma_\rho \left(\prod_{i=1}^r \frac{(u + h_i + \frac{1}{2})(u - h_i + \frac{1}{2})}{(u + h_i - \frac{1}{2})(u - h_i - \frac{1}{2})} \right),$$

where σ_ρ is the algebra automorphism given by $\sigma_\rho(h_i) = h_i + \langle \rho, \varepsilon_i \rangle$ and π_0 is the isomorphism in Theorem 3.1.

Proof. In the definition of the action of the degenerate affine BMW algebra in Theorem 2.2, y_1 acts on $M \otimes V$ as $\frac{1}{2}y + \gamma$, and

$$e_1 y_1^\ell e_1 \text{ acts on } M \otimes V^{\otimes 2} \text{ as } z_V^{(\ell)} e_1.$$

Also

$$e_1 \text{ and } y_1 \text{ in } \mathcal{W}_2 \text{ act on } M \otimes V^{\otimes 2} \text{ with } M = L(0) \otimes V^{\otimes(k-1)}$$

in the same way that

$$e_k \text{ and } y_k \text{ in } \mathcal{W}_{k+1} \text{ act on } M \otimes V^{\otimes(k+1)} \text{ with } M = L(0).$$

By Proposition 3.2, $z_{k-1}^{(\ell)} e_k = e_k y_k^\ell e_k$. Hence, as operators on $L(0) \otimes V^{\otimes(k-1)}$,

$$z_V(u) + \epsilon u - \frac{1}{2} = z_{k-1}(u) + \epsilon u - \frac{1}{2} \quad (3.5)$$

We will use (3.4) to compute the action of this operator on the $L(\mu) \otimes \mathcal{W}_{k-1}^\mu$ isotypic component in the $U\mathfrak{g} \otimes \mathcal{W}_{k-1}$ -module decomposition

$$L(0) \otimes V^{\otimes(k-1)} \cong \bigoplus_{\mu} L(\mu) \otimes \mathcal{W}_{k-1}^\mu. \quad (3.6)$$

As an operator on $L(0) \otimes V$,

$$\gamma = \frac{1}{2}(\langle \varepsilon_1, \varepsilon_1 + 2\rho \rangle - \langle \varepsilon_1, \varepsilon_1 + 2\rho \rangle + \langle 0, 0 + 2\rho \rangle) = 0 \quad \text{by (1.18),}$$

and so

$$z_0^{(\ell)} = \epsilon (\text{id} \otimes \text{tr}_V)((\frac{1}{2}y + \gamma)^\ell) = \epsilon (\text{id} \otimes \text{tr}_V)((\frac{1}{2}y)^\ell) = \epsilon \dim(V) (\frac{1}{2}y)^\ell.$$

Therefore, since $\dim(V) = \epsilon + y$,

$$z_0(u) = \sum_{\ell \in \mathbb{Z}_{\geq 0}} z_V^{(\ell)} u^{-\ell} = \sum_{\ell \in \mathbb{Z}_{\geq 0}} \epsilon \dim(V) (\frac{1}{2}y)^\ell u^{-\ell} = \epsilon \dim(V) \frac{1}{1 - \frac{1}{2}y u^{-1}} = \frac{1 + \epsilon y}{1 - \frac{1}{2}y u^{-1}}.$$

Thus, as an operator on $L(0) \otimes V$,

$$z_0(u) + \epsilon u - \frac{1}{2} = \frac{1 + \epsilon y}{1 - \frac{1}{2}y u^{-1}} + \epsilon u - \frac{1}{2} = \frac{(\epsilon u + \frac{1}{2})(u + \frac{1}{2}y)}{u - \frac{1}{2}y}. \quad (3.7)$$

By the first identity in (1.12) and the definition of Φ in Theorem 1.2,

$$y_k \in \mathcal{W}_k \text{ acts on } L(0) \otimes V^{\otimes k} = (L(0) \otimes V^{\otimes(k-1)}) \otimes V \text{ as } \frac{1}{2}y + \gamma.$$

If $L(\mu)$ is an irreducible $U\mathfrak{g}$ -module in $L(0) \otimes V^{\otimes(k-1)}$, then (1.18), (2.13), and (2.16) give that y_k acts on the $L(\lambda)$ component of $L(\mu) \otimes V$ by the constant $c(\lambda, \mu) = 0$ when $\lambda = \mu$, and by the constant

$$\begin{aligned} c(\lambda, \mu) &= \frac{1}{2}y + \frac{1}{2}(\langle \mu \pm \varepsilon_i, \mu \pm \varepsilon_i + 2\rho \rangle - \langle \mu, \mu + 2\rho \rangle - \langle \varepsilon_1, \varepsilon_1 + 2\rho \rangle) \\ &= \begin{cases} \frac{1}{2}y + c(\lambda/\mu), & \text{if } \mu \subseteq \lambda, \\ -\frac{1}{2}y - c(\mu/\lambda), & \text{if } \mu \supseteq \lambda, \end{cases} \quad \text{where } \lambda = \mu \pm \varepsilon_i. \end{aligned} \quad (3.8)$$

As in [21, Theorem 2.6], the irreducible \mathcal{W}_k -module $\mathcal{W}_k^{\mu/0} = \mathcal{W}_k^\mu$ has a basis $\{v_T\}$ indexed by *up-down tableaux* $T = (T^{(0)}, T^{(1)}, \dots, T^{(k)})$, where $T^{(0)} = \emptyset$, $T^{(k)} = \mu$, and $T^{(i)}$ is a partition obtained from $T^{(i-1)}$ by adding or removing a box (or, in some cases when $\mathfrak{g} = \mathfrak{so}_{2r+1}$ leaving the partition the same; see (2.21)) and

$$y_i v_T = \begin{cases} (\frac{1}{2}y + c(b))v_T, & \text{if } b = T^{(i)}/T^{(i-1)}, \\ (-\frac{1}{2}y - c(b))v_T, & \text{if } b = T^{(i-1)}/T^{(i)}, \\ 0, & \text{if } T^{(i-1)} = T^{(i)}. \end{cases}$$

Thus the product on the right hand side of (3.4)

$$\prod_{i=1}^{k-1} \frac{(u + y_i - 1)(u + y_i + 1)(u - y_i)^2}{(u + y_i)^2(u - y_i + 1)(u - y_i - 1)} \quad \text{acts on } L(\mu) \otimes \mathcal{W}_{k-1}^\mu \text{ in (3.6)}$$

by

$$\prod_{i=1}^{k-1} \frac{(u + c(T^{(i)}, T^{i-1}) - 1)(u + c(T^{(i)}, T^{(i-1)}) + 1)(u - c(T^{(i)}, T^{(i-1)}))^2}{(u + c(T^{(i)}, T^{i-1}))^2(u - c(T^{(i)}, T^{i-1}) + 1)(u - c(T^{(i)}, T^{i-1}) - 1)} \quad (3.9)$$

for any up-down tableau T of length k and shape μ . If a box is added (or removed) at step i and then removed (or added) at step j , then the i and j factors of this product cancel. Therefore (3.9) is equal to

$$\prod_{b \in \mu} \frac{(u + \frac{1}{2}y + c(b) - 1)(u + \frac{1}{2}y + c(b) + 1)(u - \frac{1}{2}y - c(b))^2}{(u + \frac{1}{2}y + c(b))^2(u - \frac{1}{2}y - c(b) + 1)(u - \frac{1}{2}y - c(b) - 1)} \quad (3.10)$$

(see [21, Lemma 3.8]). If $\mu = (\mu_1, \dots, \mu_r)$, simplifying one row at a time,

$$\begin{aligned} \prod_{b \in \mu} \frac{(u + \frac{1}{2}y + c(b) - 1)(u + \frac{1}{2}y + c(b) + 1)}{(u + \frac{1}{2}y + c(b))^2} &= \prod_{i=1}^r \frac{(u + \frac{1}{2}y - i)(u + \frac{1}{2}y + \mu_i - i + 1)}{(u + \frac{1}{2}y + 1 - i)(u + \frac{1}{2}y + \mu_i - i)} \\ &= \frac{u + \frac{1}{2}y - r}{u + \frac{1}{2}y} \prod_{i=1}^r \frac{(u + \frac{1}{2}y + \mu_i - i + 1)}{(u + \frac{1}{2}y + \mu_i - i)}. \end{aligned} \quad (3.11)$$

It follows that (3.10) is equal to

$$\begin{aligned} &\frac{(u + \frac{1}{2}y - r)}{(u + \frac{1}{2}y)} \frac{(u - \frac{1}{2}y)}{(u - \frac{1}{2}y + r)} \prod_{i=1}^r \frac{(u + \frac{1}{2}y + \mu_i - i + 1)}{(u + \frac{1}{2}y + \mu_i - i)} \frac{(u - \frac{1}{2}y - (\mu_i - i))}{(u - \frac{1}{2}y - (\mu_i - i + 1))} \\ &= \frac{(u + \frac{1}{2}y - r)}{(u + \frac{1}{2}y)} \frac{(u - \frac{1}{2}y)}{(u - \frac{1}{2}y + r)} \text{ev}_{\mu+\rho} \left(\prod_{i=1}^r \frac{(u + h_i + \frac{1}{2})}{(u + h_i - \frac{1}{2})} \frac{(u - h_i + \frac{1}{2})}{(u - h_i - \frac{1}{2})} \right), \end{aligned} \quad (3.12)$$

since $\text{ev}_{\mu+\rho}(h_i) = \mu_i + \rho_i = \mu_i + \frac{1}{2}(y - 2i + 1) = \frac{1}{2}y + \frac{1}{2} + \mu_i - i$.

Combining (3.7) and (3.12), the identity (3.5) gives that, as operators on $L(\mu) \otimes \mathcal{W}_{k-1}^\mu$ in (3.6),

$$z_V(u) + \epsilon u - \frac{1}{2} = (\epsilon u + \frac{1}{2}) \frac{(u + \frac{1}{2}y - r)}{(u - \frac{1}{2}y + r)} \text{ev}_{\mu+\rho} \left(\prod_{i=1}^r \frac{(u + h_i + \frac{1}{2})}{(u + h_i - \frac{1}{2})} \frac{(u - h_i + \frac{1}{2})}{(u - h_i - \frac{1}{2})} \right).$$

By Theorem 3.1, the desired result follows. \square

3.2 Central elements $Z_V^{(\ell)}$

Let $Z_0^{(\ell)}$, z and q be the parameters of the affine BMW algebra W_k . Let u be a variable and define $Z_i^{(\ell)}, Z_i^{(-\ell)} \in W_k$ for $i = 1, \dots, k-1$ by

$$\begin{aligned} Z_i^+(u) &+ \frac{z^{-1}}{q-q^{-1}} - \frac{u^2}{u^2-1} \\ &= \left(Z_0^+ + \frac{z^{-1}}{q-q^{-1}} - \frac{u^2}{u^2-1} \right) \prod_{j=1}^i \frac{(u-Y_j)^2(u-q^{-2}Y_j^{-1})(u-q^2Y_j^{-1})}{(u-Y_j^{-1})^2(u-q^2Y_j)(u-q^{-2}Y_j)}, \end{aligned} \quad (3.13)$$

$$\begin{aligned} Z_i^-(u) &- \frac{z}{q-q^{-1}} - \frac{u^2}{u^2-1} \\ &= \left(Z_0^- - \frac{z}{q-q^{-1}} - \frac{u^2}{u^2-1} \right) \prod_{j=1}^i \frac{(u-Y_j^{-1})^2(u-q^2Y_j)(u-q^{-2}Y_j)}{(u-Y_j)^2(u-q^{-2}Y_j^{-1})(u-q^2Y_j^{-1})} \end{aligned} \quad (3.14)$$

where

$$Z_i^+(u) = \sum_{\ell \in \mathbb{Z}_{\geq 0}} Z_i^{(\ell)} u^{-\ell} \quad \text{and} \quad Z_i^-(u) = \sum_{\ell \in \mathbb{Z}_{\geq 0}} Z_i^{(-\ell)} u^{-\ell} \quad \text{for } i = 0, \dots, k-1.$$

The following proposition is equivalent to [3, Lemma 7.4] and is also proved in [6, Theorem 3.6 and Remark 3.8].

Proposition 3.4. *In the affine BMW algebra W_k ,*

$$E_{i+1}Y_i^\ell E_{i+1} = Z_i^{(\ell)} E_{i+1}, \quad \text{for } i = 0, 1, \dots, k-2 \text{ and } \ell \in \mathbb{Z}.$$

The following theorem uses the identity (3.13) and the action of the affine BMW algebra on tensor space to provide a formula for the Harish-Chandra images of the central elements $Z_V^{(\ell)} = \epsilon(\text{id} \otimes \text{qtr}_V)((z\mathcal{R}_{21}\mathcal{R})^\ell)$ in the Drinfeld-Jimbo quantum group $U_{\mathfrak{h}}\mathfrak{g}$ for orthogonal and symplectic Lie algebras \mathfrak{g} . By Theorem 2.5 these central elements are natural parameters for the affine BMW algebras.

Theorem 3.5. *Let $U = U_{\mathfrak{h}}\mathfrak{g}$ be the Drinfeld-Jimbo quantum group corresponding to $\mathfrak{g} = \mathfrak{so}_{2r+1}$, \mathfrak{sp}_{2r} or \mathfrak{so}_{2r} and use notations for \mathfrak{h}^* as in (2.5)-(2.16). Identify U_0 as a subalgebra of $\mathbb{C}[L_1^{\pm 1}, \dots, L_r^{\pm 1}]$ where $\text{ev}_{\varepsilon_i}(L_j) = q^{(\varepsilon_i, \varepsilon_j)} = q^{\delta_{ij}}$ (so that $L_i = e^{\frac{1}{2}hF_{ii}}$, where F_{ii} is as in (2.8)). Let y and ϵ be as in (2.13) and (2.5), $z = \epsilon q^y$, and $V = L(\varepsilon_1)$. Let $Z_V^{(\ell)}$ be the central elements in $U_{\mathfrak{h}}\mathfrak{g}$ defined by*

$$Z_V^{(\ell)} = \epsilon(\text{id} \otimes \text{qtr}_V)((z\mathcal{R}_{21}\mathcal{R})^\ell)$$

and write

$$Z_V^+(u) = \sum_{\ell \in \mathbb{Z}_{\geq 0}} Z_V^{(\ell)} u^{-\ell} \quad \text{and} \quad Z_V^-(u) = \sum_{\ell \in \mathbb{Z}_{\geq 0}} Z_V^{(-\ell)} u^{-\ell}.$$

Then

$$\begin{aligned} &\pi_0 \left(Z_V^+(u) + \frac{z^{-1}}{q-q^{-1}} - \frac{u^2}{u^2-1} \right) \\ &= \left(\frac{z}{q-q^{-1}} \right) \frac{(u+q)(u-q^{-1})(u-\epsilon q^{2r-y})}{(u+1)(u-1)(u-\epsilon q^{y-2r})} \sigma_\rho \left(\prod_{i=1}^r \frac{(u-\epsilon L_i^{-2}q^{-1})(u-\epsilon L_i^2q^{-1})}{(u-\epsilon L_i^{-2}q)(u-\epsilon L_i^2q)} \right) \end{aligned}$$

and

$$\begin{aligned} & \pi_0 \left(Z_V^-(u) - \frac{z}{q - q^{-1}} - \frac{u^2}{u^2 - 1} \right) \\ &= -\frac{z^{-1}}{(q - q^{-1})} \frac{(u - q)(u + q^{-1})}{(u + 1)(u - 1)} \frac{(u - \epsilon q^{y-2r})}{(u - \epsilon q^{2r-y})} \sigma_\rho \left(\prod_{i=1}^r \frac{(u - \epsilon L_i^{-2} q)(u - \epsilon L_i^2 q)}{(u - \epsilon L_i^{-2} q^{-1})(u - \epsilon L_i^2 q^{-1})} \right), \end{aligned}$$

where σ_ρ is the algebra automorphism given by $\sigma_\rho(L_i) = q^{(\rho, \varepsilon_i)} L_i$ and π_0 is the isomorphism in Theorem 3.1.

Proof. In the definition of the action of the affine BMW algebra in Theorem 1.3, Y_1 acts on $M \otimes V$ as $z\mathcal{R}_{21}\mathcal{R}$, and

$$E_1 Y_1^\ell E_1 \text{ acts on } M \otimes V^{\otimes 2} \text{ as } Z_V^{(\ell)} E_1.$$

Also

$$E_1 \text{ and } Y_1 \text{ in } W_2 \text{ act on } M \otimes V^{\otimes 2} \text{ with } M = L(0) \otimes V^{\otimes(k-1)}$$

in the same way that

$$E_k \text{ and } Y_k \text{ in } W_{k+1} \text{ act on } M \otimes V^{\otimes(k+1)} \text{ with } M = L(0).$$

By Proposition 3.4, $Z_{k-1}^{(\ell)} E_k = E_k Y_k^\ell E_k$ and so it follows that, as operators on $L(0) \otimes V^{\otimes(k-1)}$,

$$Z_V^+(u) + \frac{z^{-1}}{q - q^{-1}} - \frac{u^2}{u^2 - 1} = Z_{k-1}^+(u) + \frac{z^{-1}}{q - q^{-1}} - \frac{u^2}{u^2 - 1} \quad (3.15)$$

and

$$Z_V^-(u) - \frac{z}{q - q^{-1}} - \frac{u^2}{u^2 - 1} = Z_{k-1}^-(u) - \frac{z}{q - q^{-1}} - \frac{u^2}{u^2 - 1} \quad (3.16)$$

We will use (3.13) and (3.14) to compute the action of these operators on the $L(\mu) \otimes W_{k-1}^\mu$ isotypic component in the $U_h \mathfrak{g} \otimes W_{k-1}$ -module decomposition

$$L(0) \otimes V^{\otimes(k-1)} \cong \bigoplus_{\mu} L(\mu) \otimes W_{k-1}^\mu. \quad (3.17)$$

As an operator on $L(0) \otimes V$, $z(\mathcal{R}_{21}\mathcal{R}) = zq^{(\varepsilon_1, \varepsilon_1 + 2\rho) - \langle \varepsilon_1, \varepsilon_1 + 2\rho \rangle + \langle 0, 0 + 2\rho \rangle} = z$. Hence

$$Z_V^{(\ell)} = \epsilon (\text{id} \otimes \text{qtr}_V)((z\mathcal{R}_{21}\mathcal{R})^\ell) = z^\ell \epsilon \dim_q(V).$$

Therefore, since $\epsilon \dim_q(V) = \frac{z - z^{-1}}{q - q^{-1}} + 1$,

$$Z_V^+(u) = \sum_{\ell \in \mathbb{Z}_{\geq 0}} \epsilon \dim_q(V) z^\ell u^{-\ell} = \epsilon \dim_q(V) \frac{1}{1 - zu^{-1}} = \frac{z - z^{-1} + (q - q^{-1})}{(q - q^{-1})(1 - zu^{-1})}.$$

A similar computation of Z_V^- yields

$$Z_V^-(u) = \frac{z - z^{-1} + q - q^{-1}}{(q - q^{-1})(1 - z^{-1}u^{-1})}.$$

Thus, as operators on $L(0) \otimes V$,

$$Z_V^+ + \frac{z^{-1}}{q-q^{-1}} - \frac{u^2}{u^2-1} = \frac{z}{(q-q^{-1})} \frac{(1-z^{-1}u^{-1})}{(1-zu^{-1})} \frac{(u+q)(u-q^{-1})}{(u+1)(u-1)} \quad (3.18)$$

and

$$Z_V^- - \frac{z}{q-q^{-1}} - \frac{u^2}{u^2-1} = \frac{-z^{-1}}{(q-q^{-1})} \frac{(1-zu^{-1})}{(1-z^{-1}u^{-1})} \frac{(u-q)(u+q^{-1})}{(u+1)(u-1)}. \quad (3.19)$$

By (1.30) and the definition of Φ in Theorem 1.3,

$$Y_k \in W_k \quad \text{acts on} \quad L(0) \otimes V^{\otimes k} = (L(0) \otimes V^{\otimes(k-1)}) \otimes V \quad \text{as} \quad z\mathcal{R}_{21}\mathcal{R}.$$

If $L(\mu)$ is an irreducible $U_h\mathfrak{g}$ -module in $L(0) \otimes V^{\otimes(k-1)}$, then (1.29) and (2.16) give that Y_k acts on the $L(\lambda)$ component of $L(\mu) \otimes V$ by the constant $\epsilon q^{2c(\lambda,\mu)} = 1 \cdot q^0 = 1$, when $\mathfrak{g} = \mathfrak{so}_{2r+1}$ and $\lambda = \mu$, and by the constant

$$\epsilon q^{2c(\lambda,\mu)} = \begin{cases} \epsilon q^{y+2c(\lambda/\mu)}, & \text{if } \mu \subseteq \lambda, \\ \epsilon q^{-y-2c(\mu/\lambda)}, & \text{if } \mu \supseteq \lambda, \end{cases} = \begin{cases} zq^{2c(\lambda/\mu)}, & \text{if } \mu \subseteq \lambda, \\ z^{-1}q^{-2c(\mu/\lambda)}, & \text{if } \mu \supseteq \lambda, \end{cases}, \quad \text{where } \lambda = \mu \pm \epsilon_i.$$

and $c(\lambda,\mu)$ is as computed in (3.8). As in [22, Theorem 6.3(b)], the irreducible W_k -module $W_k^{\mu/0} = W_k^\mu$ has a basis $\{v_T\}$ indexed by *up-down tableaux* $T = (T^{(0)}, T^{(1)}, \dots, T^{(k)})$, where $T^{(0)} = \emptyset$, $T^{(k)} = \mu$, and $T^{(i)}$ is a partition obtained from $T^{(i-1)}$ by adding or removing a box, and

$$Y_i v_T = \begin{cases} zq^{2c(b)} v_T, & \text{if } b = T^{(i)}/T^{(i-1)}, \\ z^{-1}q^{-2c(b)} v_T, & \text{if } b = T^{(i-1)}/T^{(i)}, \\ v_T, & \text{if } T^{(i-1)} = T^{(i)}. \end{cases}$$

Thus

$$\prod_{i=1}^{k-1} \frac{(u - Y_i)^2 (u - q^{-2}Y_i^{-1})(u - q^2Y_i^{-1})}{(u - Y_i^{-1})^2 (u - q^2Y_i)(u - q^{-2}Y_i)} \quad \text{acts on } L(\mu) \otimes W_{k-1}^\mu \text{ in (3.17)}$$

by

$$\prod_{i=1}^{k-1} \frac{(u - \epsilon q^{2c(T^{(i)}, T^{(i-1)})})^2 (u - \epsilon q^{-2}q^{-2c(T^{(i)}, T^{(i-1)})})(u - \epsilon q^2q^{-2c(T^{(i)}, T^{(i-1)})})}{(u - \epsilon q^{-2c(T^{(i)}, T^{(i-1)})})^2 (u - \epsilon q^2q^{2c(T^{(i)}, T^{(i-1)})})(u - \epsilon q^{-2}q^{2c(T^{(i)}, T^{(i-1)})})} \quad (3.20)$$

for any up-down tableau T of length k and shape μ . If a box is added (or removed) at step i and then removed (or added) at step j , then the i and j factors of this product cancel. Therefore (3.20) is equal to

$$\prod_{b \in \mu} \frac{(u - zq^{2c(b)})^2 (u - z^{-1}q^{-2(c(b)+1)})(u - z^{-1}q^{-2(c(b)-1)})}{(u - z^{-1}q^{-2c(b)})^2 (u - zq^{2(c(b)+1)})(u - zq^{2(c(b)-1)})}. \quad (3.21)$$

Simplifying one row at a time,

$$\begin{aligned} \prod_{b \in \mu} \frac{(u - z^{-1}q^{-2(c(b)-1)})(u - z^{-1}q^{-2(c(b)+1)})}{(u - z^{-1}q^{-2c(b)})(u - z^{-1}q^{-2c(b)})} &= \prod_{i=1}^r \frac{(u - z^{-1}q^{-2(-i)})(u - z^{-1}q^{-2(\mu_i - i + 1)})}{(u - z^{-1}q^{-2(-(i-1))})(u - z^{-1}q^{-2(\mu_i - i)})} \\ &= \frac{u - z^{-1}q^{2r}}{u - z^{-1}q^{2 \cdot 0}} \prod_{i=1}^r \frac{u - z^{-1}q^{-2(\mu_i - i + 1)}}{u - z^{-1}q^{-2(\mu_i - i)}} \end{aligned}$$

if $\mu = (\mu_1, \dots, \mu_r)$. It follows that (3.21) is equal to

$$\begin{aligned} & \frac{(u - z^{-1}q^{2r})}{(u - z^{-1})} \frac{(u - z)}{(u - zq^{-2r})} \prod_{i=1}^r \frac{(u - z^{-1}q^{-2(\mu_i-i+1)})}{(u - z^{-1}q^{-2(\mu_i-i)})} \cdot \frac{(u - zq^{2(\mu_i-i)})}{(u - zq^{2(\mu_i-i+1)})} \\ &= \frac{(u - \epsilon q^{-(y-2r)})}{(u - z^{-1})} \frac{(u - z)}{(u - \epsilon q^{y-2r})} \text{ev}_{\mu+\rho} \left(\prod_{i=1}^r \frac{(u - \epsilon L_i^{-2}q^{-1})(u - \epsilon L_i^2q^{-1})}{(u - \epsilon L_i^{-2}q)(u - \epsilon L_i^2q)} \right), \end{aligned} \quad (3.22)$$

since $z^{-1}q^{2r} = \epsilon q^{-y}q^{2r} = \epsilon q^{2r-y}$ and

$$\text{ev}_{\mu+\rho}(L_i^2) = q^{\langle \mu+\rho, 2\epsilon_i \rangle} = q^{2\mu_i+(y-2i+1)} = q^{y+1+2(\mu_i-i)} = \epsilon z q^{2(\mu_i-i)+1}.$$

Combining (3.18) and (3.22), the identity (3.13) gives that, as operators on $L(\mu) \otimes W_{k-1}^\mu$ in (3.17),

$$\begin{aligned} & Z_k^+ + \frac{z^{-1}}{q - q^{-1}} - \frac{u^2}{u^2 - 1} \\ &= \frac{z}{(q - q^{-1})} \frac{(u + q)(u - q^{-1})}{(u + 1)(u - 1)} \frac{(u - \epsilon q^{2r-y})}{(u - \epsilon q^{y-2r})} \text{ev}_{\mu+\rho} \left(\prod_{i=1}^r \frac{(u - \epsilon L_i^{-2}q^{-1})(u - \epsilon L_i^2q^{-1})}{(u - \epsilon L_i^{-2}q)(u - \epsilon L_i^2q)} \right). \end{aligned} \quad (3.23)$$

Similarly, $Z_k^- - \frac{z}{q - q^{-1}} + \frac{1}{u^2 - 1}$ acts on the $L(\mu) \otimes W_{k-1}^\mu$ isotypic component in the $U_h \mathfrak{g} \otimes W_{k-1}$ -module decomposition in (3.17) by

$$- \frac{z^{-1}}{(q - q^{-1})} \frac{(u - q)(u + q^{-1})}{(u + 1)(u - 1)} \frac{(u - \epsilon q^{2r-y})}{(u - \epsilon q^{y-2r})} \text{ev}_{\mu+\rho} \left(\prod_{i=1}^r \frac{(u - \epsilon L_i^{-2}q)(u - \epsilon L_i^2q)}{(u - \epsilon L_i^{-2}q^{-1})(u - \epsilon L_i^2q^{-1})} \right).$$

By Theorem 3.1, the desired results follow. \square

In the following corollary, we shall repackage Theorem 3.5 to give a formula for the Harish-Chandra image of $Z_V^{(\ell)}$ in terms of ‘‘Weyl characters’’. To do this we will use the universal characters of [17] following the notation in [16, §6]. For a formal alphabet $Y = \{y_1, y_2, \dots\}$ let $sa_\lambda(Y)$ be the universal Weyl character for \mathfrak{gl}_r , $sp_\lambda(Y)$ the universal Weyl character for \mathfrak{sp}_{2r} , and $so_\lambda(Y)$ the universal Weyl character for the orthogonal cases.

Let $X = \{x_1, x_2, \dots\}$ be a second formal alphabet. The Cauchy-Littlewood identities (see [17, Lemma 1.5.1], [27, Theorems 7.8FG and 7.9C], and [16, (6.4) and (6.5)]) are

$$\begin{aligned} & \prod_{i,j} \frac{1}{1 - x_i y_j} = \Omega(XY) = \sum_{\lambda} sa_\lambda(X) sa_\lambda(Y), \\ & \prod_{i \leq j} \frac{1}{1 - y_i y_j} \prod_{i,j} \frac{1}{1 - x_i y_j} = \Omega(XY - sa_{(2)}(Y)) = \sum_{\lambda} sa_\lambda(Y) so_\lambda(X), \\ & \prod_{i < j} \frac{1}{1 - y_i y_j} \prod_{i,j} \frac{1}{1 - x_i y_j} = \Omega(XY - sa_{(1^2)}(Y)) = \sum_{\lambda} sa_\lambda(Y) sp_\lambda(X), \end{aligned}$$

where Ω is the Cauchy kernel (see [16, (6.3)]). The identity [16, Lemma 6.7(a)] states

$$sa_\lambda((q - q^{-1})u^{-1}) = \begin{cases} (q - q^{-1})u^{-\ell}(-q^{-1})^{\ell-m}q^{m-1}, & \text{if } \lambda = (m, 1^{\ell-m}), \\ 0, & \text{otherwise.} \end{cases} \quad (3.24)$$

Corollary 3.6. *In the same setting as in Theorem 3.5, let y and ϵ be as in (2.13) and (2.5), $z = \epsilon q^y$, and $V = L(\epsilon_1)$. Let $Z_V^{(\ell)}$ be the central elements in the Drinfeld-Jimbo quantum group $U_{\hbar} \mathfrak{g}$ which are given by $Z_V^{(\ell)} = \epsilon(\text{id} \otimes \text{qtr}_{L(\epsilon_1)})((z\mathcal{R}_{21}\mathcal{R})^\ell)$. Let X be the formal alphabet given by $X = \sum_{i \in \hat{V}} L_i^2$ and fix $c = 1$ if ℓ is even and $c = 0$ if ℓ is odd. Then for $\ell \geq 1$,*

$$\pi_0(Z_V^{(\ell)}) = \sigma_\rho \left(c + z\epsilon^\ell \sum_{m=1}^{\ell} (q - q^{-1})(-1)^{\ell-m} q^{-(\ell-2m+1)} s_{(m,1^{\ell-m})}(X) \right)$$

where $s_{(m,1^{\ell-m})}(X) = so_{(m,1^{\ell-m})}(X)$ in the orthogonal cases and $s_{(m,1^{\ell-m})}(X) = sp_{(m,1^{\ell-m})}(X)$ in the symplectic case.

Proof. Let \hat{V} as in (2.6), $L_{-i} = L_i^{-1}$ where L_i is as in the statement of Theorem 3.5, and let $L_{\epsilon_0} = 1$. The identity in Theorem 3.5 can be rewritten as

$$\pi_0 \left(Z_V^+(u) + \frac{z^{-1}}{q - q^{-1}} - \frac{u^2}{u^2 - 1} \right) = \sigma_\rho \left(\frac{z}{q - q^{-1}} \frac{u^2 - q^{2\epsilon}}{u^2 - 1} \prod_{j \in \hat{V}} \frac{(1 - L_{\text{wt}(v_j)}^2 q^{-1}(\epsilon u)^{-1})}{(1 - L_{\text{wt}(v_j)}^2 q(\epsilon u)^{-1})} \right). \quad (3.25)$$

By (3.24),

$$sa_{(2)}((q - q^{-1})(\epsilon u)^{-1}) = (q^2 - 1)(\epsilon u)^{-2} \quad \text{and} \quad sa_{(12)}((q - q^{-1})(\epsilon u)^{-1}) = (q^{-2} - 1)(\epsilon u)^{-2}.$$

So the Cauchy-Littlewood identities give

$$\begin{aligned} & \frac{(1 - q^2(\epsilon u)^{-2})}{1 - (\epsilon u)^{-2}} \prod_{i \in \hat{V}} \frac{(1 - L_i^2 q^{-1}(\epsilon u)^{-1})}{(1 - L_i^2 q(\epsilon u)^{-1})} \\ &= \Omega(X(q - q^{-1})(\epsilon u)^{-1} - sa_{(2)}((q - q^{-1})(\epsilon u)^{-1})) = \sum_{\lambda} sa_{\lambda}((q - q^{-1})(\epsilon u)^{-1}) so_{\lambda}(X) \\ &= \sum_{\ell \in \mathbb{Z}_{\geq 0}} \left(\sum_{m=1}^{\ell} (q - q^{-1})(-q^{-1})^{\ell-m} q^{m-1} so_{(m,1^{\ell-m})}(X) \right) (\epsilon u)^{-\ell} \\ &= \sum_{\ell \in \mathbb{Z}_{\geq 0}} \epsilon^{\ell} (q - q^{-1}) \left(\sum_{m=1}^{\ell} (q - q^{-1})(-1)^{\ell-m} q^{-(\ell-2m+1)} so_{(m,1^{\ell-m})}(X) \right) u^{-\ell} \end{aligned}$$

in the orthogonal case, and

$$\begin{aligned} & \frac{(1 - q^{-2}(\epsilon u)^{-2})}{1 - (\epsilon u)^{-2}} \prod_{i \in \hat{V}} \frac{(1 - L_i^2 q^{-1}(\epsilon u)^{-1})}{(1 - L_i^2 q(\epsilon u)^{-1})} \\ &= \Omega(X(q - q^{-1})(\epsilon u)^{-1} - sa_{(12)}((q - q^{-1})(\epsilon u)^{-1})) = \sum_{\lambda} sa_{\lambda}((q - q^{-1})(\epsilon u)^{-1}) sp_{\lambda}(X) \\ &= \sum_{\ell \in \mathbb{Z}_{\geq 0}} \left(\sum_{m=1}^{\ell} (q - q^{-1})(-q^{-1})^{\ell-m} q^{m-1} sp_{(m,1^{\ell-m})}(X) \right) (\epsilon u)^{-\ell} \\ &= \sum_{\ell \in \mathbb{Z}_{\geq 0}} \epsilon^{\ell} (q - q^{-1}) \left(\sum_{m=1}^{\ell} (-1)^{\ell-m} q^{-(\ell-2m+1)} sp_{(m,1^{\ell-m})}(X) \right) u^{-\ell} \end{aligned}$$

in the symplectic case. The statement now follows by noting that $u^2/(u^2 - 1) = 1/(1 - u^{-2}) = \sum_{k \in \mathbb{Z}_{\geq 0}} u^{-2k}$ and taking the coefficient of $u^{-\ell}$ on each side of (3.25). \square

4 Symplectic and orthogonal higher Casimir elements

Our final goal in this paper will be to connect the central elements appearing naturally as parameters of the affine and degenerate affine BMW algebras (see Theorems 2.2 and 2.5) to higher Casimir elements for orthogonal and symplectic Lie algebras and quantum groups. In the degenerate case, we explain how the generating function for $z_V^{(\ell)}$ derived in Theorem 3.3 can be matched up with the generating functions for central elements given by Perelomov-Popov in [23, 24]. Expositions of the Perelomov-Popov results are also in [19, §7.1] and [31, §127]. In the affine case we show how the formula for $Z_V^{(\ell)}$ in Corollary 3.6 can be derived as a special case of a remarkable identity for central elements in quantum groups discovered by Baumann [2, Thm. 1].

4.1 The central elements $z_V^{(\ell)}$ as higher Casimir elements

Returning to the notation developed in the preliminaries of Section 2, let $\mathfrak{g} = \mathfrak{gl}_r$ with nondegenerate ad-invariant form $\langle \cdot, \cdot \rangle$ as in (2.1) and operator $\gamma = \gamma^{\mathfrak{gl}}$ as in (2.2). Then

$$(\text{id} \otimes \text{tr}_V)(\gamma^\ell) = \sum_{i_1, i_2, \dots, i_\ell} E_{i_1 i_2} E_{i_2 i_3} \cdots E_{i_\ell i_1}$$

are the central elements of $U\mathfrak{gl}_n$ found, for example, in Gelfand [8, (3)]. Perelomov-Popov [23, 24] generalized this construction to $\mathfrak{g} = \mathfrak{so}_{2r+1}$, \mathfrak{sp}_{2r} , and \mathfrak{so}_{2r} , by letting F_{ij} be the natural spanning set for \mathfrak{g} given in (2.7), viewing $F = (F_{ij})_{i,j \in \hat{V}}$ as a matrix with entries in \mathfrak{g} , and writing

$$\text{tr} F^k = \sum_{i_1, i_2, \dots, i_k \in \hat{V}} F_{i_1 i_2} F_{i_2 i_3} \cdots F_{i_k i_1} \quad \text{for } k \in \mathbb{Z}_{>0}, \quad (4.1)$$

as an element of the enveloping algebra $U\mathfrak{g}$ (see [19, Thm. 7.1.7]). These elements are central in $U\mathfrak{g}$ and Perelomov-Popov gave the following generating function formula for their Harish-Chandra images (see [31, §127]). The proof we give below shows that the result of Perelomov-Popov is equivalent to Theorem 3.3 (which we obtained from the degenerate affine BMW algebra and Schur-Weyl duality). A proof of Theorem 4.1 using the theory of twisted Yangians is given in [19, §7.1].

Theorem 4.1. *(Perelomov-Popov) [19, Cor. 7.1.8] Let $\mathfrak{g} = \mathfrak{so}_{2r+1}$ or \mathfrak{sp}_{2r} or \mathfrak{so}_{2r} , use notations for \mathfrak{h}^* as in Section 2 and let h_1, \dots, h_r be the basis of \mathfrak{h} dual to the orthonormal basis $\varepsilon_1, \dots, \varepsilon_r$ of \mathfrak{h}^* . Let $\rho'_r = \frac{1}{2} - \frac{1}{2}y$, $l_0 = 0$ in the case that $\mathfrak{g} = \mathfrak{so}_{2r+1}$ and let*

$$l_i = -l_{-i} = h_i + \rho_i, \quad \text{for } i = 1, 2, \dots, r, \quad \text{where } \rho_i = \frac{1}{2}(y - 2i + 1).$$

Then

$$\pi_0 \left(1 + \frac{x + \frac{1}{2}}{x + \frac{1}{2} - \frac{1}{2}\epsilon} \left(\sum_{\ell \in \mathbb{Z}_{\geq 0}} \frac{(-1)^\ell \text{tr}(F^\ell)}{(x + \rho'_r)^{\ell+1}} \right) \right) = \prod_{i \in \hat{V}} \frac{x + l_i + 1}{x + l_i}. \quad (4.2)$$

Proof. By (2.11),

$$\gamma = \frac{1}{2} \sum_{i,j \in \hat{V}} F_{ij} \otimes F_{ji}.$$

Let $\eta: \mathfrak{g} \rightarrow \text{End}(V)$ be the defining representation. Since $F_{-j,-i} = -\theta_{ij}F_{ij}$,

$$\begin{aligned} (\text{id} \otimes \eta)(\gamma) &= \frac{1}{2} \sum_{i,j \in \hat{V}} F_{ij} \otimes (E_{ji} - \theta_{ji}E_{-i,-j}) = \frac{1}{2} \sum_{i,j \in \hat{V}} (F_{ij} \otimes E_{ji} - \theta_{ji}F_{-j,-i} \otimes E_{ji}) \\ &= \frac{1}{2} \sum_{i,j \in \hat{V}} (F_{ij} - \theta_{ji}F_{-j,-i}) \otimes E_{ji} = \frac{1}{2} \sum_{i,j \in \hat{V}} (F_{ij} + F_{ij}) \otimes E_{ji} \\ &= \sum_{i,j \in \hat{V}} F_{ij} \otimes E_{ji} = F^t = -\theta F \theta, \quad \text{where } \theta = \begin{pmatrix} \epsilon \text{id} & 0 \\ 0 & \text{id} \end{pmatrix}. \end{aligned}$$

Thus

$$(\text{id} \otimes \text{tr}_V)(\gamma^k) = \text{tr}((F^t)^k) = \text{tr}((-\theta F \theta)^k) = (-1)^k \text{tr}(\theta^2 F^k) = (-1)^k \text{tr}(F^k), \quad (4.3)$$

which provides the connection of the elements $z_0^{(\ell)}$ appearing in Theorems 2.2 and 3.3 to the elements in (4.1).

In order to transform the generating function for the elements $(\text{id} \otimes \text{tr}_V)(\gamma^\ell)$ into the generating function for the elements $(\text{id} \otimes \text{tr}_V)((\frac{1}{2}y + \gamma)^\ell)$, notice

$$\begin{aligned} \sum_{\ell \in \mathbb{Z}_{\geq 0}} (\text{id} \otimes \text{tr}_V)((\frac{1}{2}y + \gamma)^\ell) u^{-\ell} &= (\text{id} \otimes \text{tr}_V) \left(\frac{1}{1 - (\frac{1}{2}y + \gamma)u^{-1}} \right) \\ &= (\text{id} \otimes \text{tr}_V) \left(\frac{1}{1 - \frac{1}{2}yu^{-1} - \gamma u^{-1}} \right) = (\text{id} \otimes \text{tr}_V) \left(\left(\frac{1}{1 - \frac{1}{2}yu^{-1}} \right) \left(\frac{1}{1 - \gamma \frac{u^{-1}}{1 - \frac{1}{2}yu^{-1}}} \right) \right) \\ &= (\text{id} \otimes \text{tr}_V) \left(\frac{1}{1 - \frac{1}{2}yu^{-1}} \sum_{\ell \in \mathbb{Z}_{\geq 0}} \frac{\gamma^\ell u^{-\ell}}{(1 - \frac{1}{2}yu^{-1})^\ell} \right) = u \left(\sum_{\ell \in \mathbb{Z}_{\geq 0}} \frac{(\text{id} \otimes \text{tr}_V)(\gamma^\ell)}{(u - \frac{1}{2}y)^{\ell+1}} \right) \\ &= u \left(\sum_{\ell \in \mathbb{Z}_{\geq 0}} \frac{(\text{id} \otimes \text{tr}_V)(\gamma^\ell)}{(u - \frac{1}{2} + \rho'_r)^{\ell+1}} \right), \quad \text{where } \rho'_r = \frac{1}{2} - \frac{1}{2}y. \end{aligned}$$

Then Theorem 3.3 is equivalent to

$$\begin{aligned} \pi_0 \left(1 + \frac{\epsilon u}{\epsilon u - \frac{1}{2}} \left(\sum_{\ell \in \mathbb{Z}_{\geq 0}} \frac{(\text{id} \otimes \text{tr}_V)(\gamma^\ell)}{(u - \frac{1}{2} + \rho'_r)^{\ell+1}} \right) \right) &= \pi_0 \left(1 + \frac{\epsilon}{\epsilon u - \frac{1}{2}} \sum_{\ell \in \mathbb{Z}_{\geq 0}} (\text{id} \otimes \text{tr}_V) \left((\frac{1}{2}y + \gamma)^\ell \right) u^{-\ell} \right) \\ &= \frac{(\epsilon u + \frac{1}{2})(u + \frac{1}{2}y - r)}{(\epsilon u - \frac{1}{2})(u - \frac{1}{2}y + r)} \sigma_\rho \left(\prod_{i=1}^r \frac{(u + h_i + \frac{1}{2})(u - h_i + \frac{1}{2})}{(u + h_i - \frac{1}{2})(u - h_i - \frac{1}{2})} \right) \\ &= \frac{(\epsilon u + \frac{1}{2})(u + \frac{1}{2}y - r)}{(\epsilon u - \frac{1}{2})(u - \frac{1}{2}y + r)} \left(\prod_{i=1}^r \frac{(u + h_i + \rho_i + \frac{1}{2})(u - h_i - \rho_i + \frac{1}{2})}{(u + h_i + \rho_i - \frac{1}{2})(u - h_i - \rho_i - \frac{1}{2})} \right) \\ &= \frac{(u - \frac{1}{2} + \frac{1}{2}\epsilon + \frac{1}{2})(u - \frac{1}{2} + \frac{1}{2}y - r + \frac{1}{2})}{(u - \frac{1}{2} - \frac{1}{2}\epsilon + \frac{1}{2})(u - \frac{1}{2} - \frac{1}{2}y + r + \frac{1}{2})} \left(\prod_{i=1}^r \frac{(u - \frac{1}{2} + h_i + \rho_i + 1)(u - \frac{1}{2} - h_i - \rho_i + 1)}{(u - \frac{1}{2} + h_i + \rho_i)(u - \frac{1}{2} - h_i - \rho_i)} \right). \end{aligned}$$

Replacing $x = u - \frac{1}{2}$,

$$\begin{aligned} \pi_0 & \left(1 + \frac{x + \frac{1}{2}}{x + \frac{1}{2} - \frac{1}{2}\epsilon} \left(\sum_{\ell \in \mathbb{Z}_{\geq 0}} \frac{(\text{id} \otimes \text{tr}_V)(\gamma^\ell)}{(x + \rho'_r)^{\ell+1}} \right) \right) \\ & = \frac{(x + \frac{1}{2}\epsilon + \frac{1}{2})(x + \frac{1}{2}y - r + \frac{1}{2})}{(x - \frac{1}{2}\epsilon + \frac{1}{2})(x - \frac{1}{2}y + r + \frac{1}{2})} \left(\prod_{i=1}^r \frac{(x + h_i + \rho_i + 1)(x - h_i - \rho_i + 1)}{(x + h_i + \rho_i)(x - h_i - \rho_i)} \right) \end{aligned}$$

Since

$$\frac{(x + \frac{1}{2}\epsilon + \frac{1}{2})(x + \frac{1}{2}y - r + \frac{1}{2})}{(x - \frac{1}{2}\epsilon + \frac{1}{2})(x - \frac{1}{2}y + r + \frac{1}{2})} = \begin{cases} \frac{x+l_0+1}{x+l_0}, & \text{if } \mathfrak{g} = \mathfrak{so}_{2r+1}, \\ 1, & \text{if } \mathfrak{g} = \mathfrak{sp}_{2r} \text{ or } \mathfrak{so}_{2r}, \end{cases}$$

it follows that

$$\pi_0 \left(1 + \frac{x + \frac{1}{2}}{x + \frac{1}{2} - \frac{1}{2}\epsilon} \left(\sum_{\ell \in \mathbb{Z}_{\geq 0}} \frac{(\text{id} \otimes \text{tr}_V)(\gamma^\ell)}{(x + \rho'_r)^{\ell+1}} \right) \right) = \prod_{i \in \hat{V}} \frac{x + l_i + 1}{x + l_i}.$$

In combination with (4.3), this demonstrates the equivalence of the Perelomov-Popov theorem and Theorem 3.3. \square

Remark 4.2. Since π_0 and $\rho = \rho_1 \varepsilon_1 + \cdots + \rho_n \varepsilon_n$ are defined via a specific choice of positive roots, each side of (4.2) depends on that choice, though the identity does not. The preferred choice of positive roots in [19, p. 139] differs from our preferred choice in (2.12) by the action of the Weyl group element $w = (1, -r)(2, -(r-1)) \cdots (r-1, -2)(r, -1)$, in cycle notation.

4.2 The central elements $Z_V^{(\ell)}$ as quantum higher Casimir elements

In this section, we show how the formula for the central elements $Z_V^{(\ell)}$ in Corollary 3.6 is related to an identity for central elements in quantum groups discovered by Baumann in [2, Thm. 1]. To do this we rewrite the Baumann identity for $\mathfrak{g} = \mathfrak{sp}_{2r}$, \mathfrak{so}_{2r+1} and \mathfrak{so}_{2r} and $\lambda = \varepsilon_1$ in terms of Weyl characters indexed by partitions. Then a theorem of Turaev and Wenzl computing $(\text{id} \otimes \text{qtr}_{L(\nu)})(\mathcal{R}_{21}\mathcal{R})$ provides a conversion between the expansion in Corollary 3.6 and the expansion obtained from Baumann's identity.

For $\lambda \in \mathfrak{h}^*$ define the *Weyl character*

$$s_\lambda = \frac{a_{\lambda+\rho}}{a_\rho}, \quad \text{where } a_\mu = \sum_{w \in W_0} \det(w) e^{w\mu}$$

with W_0 the Weyl group of \mathfrak{g} and $\det: W_0 \rightarrow \{\pm 1\}$ the sign character of W_0 . The expressions s_λ and a_μ are elements of the group algebra of \mathfrak{h}^* , $\mathbb{C}[\mathfrak{h}^*] = \mathbb{C}\text{-span}\{e^\nu \mid \nu \in \mathfrak{h}^*\}$ with $e^\mu e^\nu = e^{\mu+\nu}$. If $w \in W_0$ then

$$a_{w\mu} = \det(w) a_\mu \quad \text{and} \quad s_{w \bullet \mu} = \det(w) s_\mu, \quad (4.4)$$

where the *dot action* of W_0 on \mathfrak{h}^* is given by

$$w \bullet \mu = w(\mu + \rho) - \rho, \quad \text{for } w \in W_0, \mu \in \mathfrak{h}^*. \quad (4.5)$$

The W_0 -invariants in $\mathbb{C}[\mathfrak{h}^*]$ are $\mathbb{C}[\mathfrak{h}^*]^{W_0} = \mathbb{C}\text{-span}\{s_\lambda \mid \lambda \text{ dominant integral}\}$. For $\ell \in \mathbb{Z}_{\geq 0}$ let

$$\begin{aligned} \Psi_\ell: \quad \mathbb{C}[\mathfrak{h}^*]^{W_0} & \longrightarrow Z(U_{\mathfrak{h}\mathfrak{g}}) \\ s_\nu & \longmapsto (\text{id} \otimes \text{qtr}_{L(\nu)})(\mathcal{R}_{21}\mathcal{R})^\ell. \end{aligned} \quad (4.6)$$

By [7, Prop. 1.2], the map Ψ_ℓ is a vector space isomorphism.

Theorem 4.3. [2, Thm. 1] For $\ell \in \mathbb{Z}_{\geq 0}$ define

$$m_\lambda^{(\ell)} = \sum_{w \in W_0} q^{2\ell \langle w\lambda, \rho \rangle} s_{w\lambda}. \quad \text{Then } \Psi_\ell(m_\lambda^{(\ell)}) = \Psi_1(m_{\ell\lambda}^{(1/\ell)}).$$

Corollary 4.4. In the same setting as in Theorem 3.5, let y and ϵ be as in (2.13) and (2.5), $z = \epsilon q^y$, and $V = L(\varepsilon_1)$. Let $Z_V^{(\ell)}$ be the central elements in the Drinfeld-Jimbo quantum group $U_h \mathfrak{g}$ which are given by $Z_V^{(\ell)} = \epsilon(\text{id} \otimes \text{qtr}_{L(\varepsilon_1)})(z\mathcal{R}_{21}\mathcal{R})^\ell$. Then, for $\ell \geq 1$,

$$Z_V^{(\ell)} = \epsilon^\ell \Psi_1 \left(\begin{array}{c} c + z \sum_{m=\max(\ell-r+1, 1)}^{\ell} (-1)^{\ell-m} q^{-(\ell-2m+1)} s_{(m, 1^{\ell-m+1-1})} \\ + z \sum_{m=\max(y-\ell+1, 1)}^{\ell-y+r} q^{-(\ell-2m+1)} (-1)^{m-\ell+y} s_{(m, 1^{m-\ell+y-1})} \end{array} \right),$$

where c is given by

$$c = \begin{cases} 1 & \text{if } \ell \text{ is even and } \ell < y, \\ & \text{or } \ell \geq y \text{ and } \mathfrak{g} = \mathfrak{so}_{2r+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The Weyl group W_0 for $\mathfrak{g} = \mathfrak{so}_{2r+1}$ or $\mathfrak{g} = \mathfrak{sp}_{2r}$ is the group of signed permutations. With positive roots as in (2.12), the simple reflections are $s_i = (i, i+1)$ (the transposition switching ε_i and ε_{i+1} , for $i = 1, 2, \dots, r-1$) and $s_r = (r, -r)$ (the transposition switching the sign of ε_r). For $\mathfrak{g} = \mathfrak{so}_{2r}$, the Weyl group W_0 consists of signed permutations with an even number of signs, with simple reflections $s_i = (i, i+1)$ for $i = 1, 2, \dots, r-1$, and $s_r = (r-1, -r)(r, -(r-1))$.

To prove the desired identity, we will use the second identity in (4.4) to relabel the Weyl characters $s_{w\lambda}$ appearing in

$$m_{\varepsilon_1}^{(\ell)} = \sum_{w \in W_0} q^{2\ell \langle w\varepsilon_1, \rho \rangle} s_{w\varepsilon_1} \quad \text{and} \quad m_{\ell\varepsilon_1}^{(1/\ell)} = \sum_{w \in W_0} q^{2\langle w\varepsilon_1, \rho \rangle} s_{w\ell\varepsilon_1}$$

by dominant integral weights. By (2.13), $\rho_{i+1} = \rho_i - 1$ and $\rho_r = \frac{1}{2}(y - 2r + 1)$. So if $\mu = \mu_1\varepsilon_1 + \dots + \mu_r\varepsilon_r$ then

$$s_i \bullet \mu = \mu_1\varepsilon_1 + \dots + \mu_{i-1}\varepsilon_{i-1} + (\mu_{i+1} - 1)\varepsilon_i + (\mu_i + 1)\varepsilon_{i+1} + \mu_{i+2}\varepsilon_{i+2} + \dots + \mu_r\varepsilon_r$$

for $i = 1, 2, \dots, r-1$, and

$$s_r \bullet \mu = \mu_1\varepsilon_1 + \dots + \mu_{r-1}\varepsilon_{r-1} + (-\mu_r - (y - 2r + 1))\varepsilon_r, \quad \text{if } \mathfrak{g} = \mathfrak{so}_{2r+1} \text{ or } \mathfrak{sp}_{2r}, \text{ and}$$

$$s_r \bullet \mu = \mu_1\varepsilon_1 + \dots + \mu_{r-1}\varepsilon_{r-2} + (-\mu_r - 1)\varepsilon_{r-1} + (-\mu_{r-1} - 1)\varepsilon_r, \quad \text{if } \mathfrak{g} = \mathfrak{so}_{2r}.$$

In particular, $s_{\ell\varepsilon_i} = 0$ if $0 < \ell < i$, and

$$s_{\ell\varepsilon_i} = s_{s_1 s_2 \dots s_{i-1}} \bullet \ell\varepsilon_i = (-1)^{i-1} s_{(\ell-i+1)\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_i} = (-1)^{i-1} s_{(\ell-i+1, 1^{i-1})} \quad \text{if } 0 < i \leq \ell. \quad (4.7)$$

Furthermore, if $\mathfrak{g} = \mathfrak{so}_{2r+1}$ or \mathfrak{sp}_{2r} , then

$$\begin{aligned} (s_i \dots s_{r-1} s_r s_{r-1} \dots s_i) \bullet (-\ell\varepsilon_i) &= (s_i \dots s_{r-1} s_r) \bullet (-\varepsilon_i + \dots + \varepsilon_{r-1}) - (\ell - (r - i))\varepsilon_r \\ &= (s_i \dots s_{r-1}) \bullet (-\varepsilon_i + \dots + \varepsilon_{r-1}) + (\ell - (r - i) - (y - 2r + 1))\varepsilon_r \\ &= (\ell - (r - i) - (y - 2r + 1) - (r - i))\varepsilon_i = (\ell + 2i - y - 1)\varepsilon_i. \end{aligned} \quad (4.8)$$

Similarly, if $\mathfrak{g} = \mathfrak{so}_{2r}$, then

$$\begin{aligned}
(s_i \cdots s_{r-2} s_r s_{r-1} \cdots s_i) \bullet (-\ell \varepsilon_i) &= (s_i \cdots s_{r-2} s_r) \bullet (-(\varepsilon_i + \cdots + \varepsilon_{r-1}) - (\ell - (r - i)) \varepsilon_r) \\
&= (s_i \cdots s_{r-2}) \bullet (-(\varepsilon_i + \cdots + \varepsilon_{r-2}) + (\ell - (r - i) - 1) \varepsilon_{r-1}) \\
&= (\ell - (r - i) - 1 - (r - i - 1)) \varepsilon_i = (\ell + 2i - y - 1) \varepsilon_i.
\end{aligned} \tag{4.9}$$

So, letting W_{ε_1} be the stabilizer of ε_1 , $|W_{\varepsilon_1}| = 2^{r-1}(r-1)!$, and combining (4.7) and (4.8) gives

$$\begin{aligned}
\frac{1}{|W_{\varepsilon_1}|} m_{\varepsilon_1}^{(\ell)} &= \sum_{i=1}^r q^{2\ell \langle \varepsilon_i, \rho \rangle} s_{\varepsilon_i} + q^{2\ell \langle -\varepsilon_i, \rho \rangle} s_{-\varepsilon_i} = \begin{cases} q^{2\ell \langle \varepsilon_1, \rho \rangle} s_{\varepsilon_1} + q^{-2\ell \langle \varepsilon_r, \rho \rangle} s_{\varepsilon_{-r}} & \text{if } \mathfrak{g} = \mathfrak{so}_{2r+1}, \\ q^{2\ell \langle \varepsilon_1, \rho \rangle} s_{\varepsilon_1} & \text{if } \mathfrak{g} = \mathfrak{sp}_{2r} \text{ or } \mathfrak{g} = \mathfrak{so}_{2r}. \end{cases} \\
&= q^{\ell(y-1)} s_{\varepsilon_1} - a \quad \text{where } a = \begin{cases} q^{-\ell} s_0 & \text{if } \mathfrak{g} = \mathfrak{so}_{2r+1} \\ 0 & \text{otherwise,} \end{cases}
\end{aligned} \tag{4.10}$$

since $1 + 2i - y - 1 = 0$ has a solution exactly if $i = r$ and $\mathfrak{g} = \mathfrak{so}_{2r+1}$. If $\ell > 0$ then using

$$\det(s_i \cdots s_{r-1} s_r s_{r-1} \cdots s_i) = -1 = -\det(s_i \cdots s_{r-2} s_r s_{r-1} \cdots s_i),$$

$\ell + 2i - y - 1 - (i - 1) = \ell - y + i$, and (4.7), equations (4.8) and (4.9) give

$$q^{2\ell \langle -\varepsilon_i, \rho \rangle} s_{-\varepsilon_i} = \begin{cases} -q^{-\ell} s_0, & \text{if } \ell = y + 1 - 2i \text{ and } \mathfrak{g} = \mathfrak{so}_{2r+1} \text{ or } \mathfrak{sp}_{2r}, \\ q^{-\ell} s_0, & \text{if } \ell = y + 1 - 2i \text{ and } \mathfrak{g} = \mathfrak{so}_{2r}, \\ (-1)^i q^{-(y-2i+1)} s_{(\ell-y+i, 1^{i-1})}, & \text{if } \ell - y + i \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\ell - y + i \geq 1$ when $i \geq y - \ell + 1$,

$$\begin{aligned}
\frac{1}{|W_{\varepsilon_1}|} m_{\ell \varepsilon_1}^{(1/\ell)} &= \sum_{i=1}^r q^{2\ell \langle \varepsilon_i, \rho \rangle} s_{\ell \varepsilon_i} + q^{2\ell \langle -\varepsilon_i, \rho \rangle} s_{-\ell \varepsilon_i} \\
&= \left(\sum_{i=1}^{\min(\ell, r)} q^{y-2i+1} (-1)^{i-1} s_{(\ell-i+1, 1^{i-1})} \right) - b + \left(\sum_{i=\max(y-\ell+1, 1)}^r q^{-(y-2i+1)} (-1)^i s_{(\ell-y+i, 1^{i-1})} \right)
\end{aligned}$$

where $b = 0$ if $\ell \geq y$, and if $\ell < y$ then b is given by

	ℓ is odd	ℓ is even
$\mathfrak{g} = \mathfrak{so}_{2r+1}$	$q^{-\ell} s_0$	0
$\mathfrak{g} = \mathfrak{sp}_{2r}$	0	$q^{-\ell} s_0$
$\mathfrak{g} = \mathfrak{so}_{2r}$	0	$-q^{-\ell} s_0$

(so that b is nonzero exactly when $\ell = y + 1 - 2i$ has a solution with $1 \leq i \leq r$). Then reindexing (with $m = \ell - i + 1$ in the first sum and $m = \ell - y + i$ in the second sum) gives

$$\begin{aligned}
\frac{1}{|W_{\varepsilon_1}|} m_{\ell \varepsilon_1}^{(1/\ell)} &= -b + \sum_{m=\max(\ell-r+1, 1)}^{\ell} (-1)^{\ell-m} q^{y-2(\ell-m)-1} s_{(m, 1^{\ell-m+1-1})} \\
&\quad + \sum_{m=\max(y-\ell+1, 1)}^{\ell-y+r} q^{y-2(\ell-m)-1} (-1)^{m-\ell+y} s_{(m, 1^{m-\ell+y-1})}.
\end{aligned}$$

Notice that the last sum appears only if $\ell > y - r$.

Theorem 4.3 applied in the case that $\lambda = \varepsilon_1$ gives

$$\begin{aligned}
Z_V^{(\ell)} &= \epsilon(\text{id} \otimes \text{qtr}_V)((z\mathcal{R}_{21}\mathcal{R})^\ell) = \epsilon z^\ell \Psi_\ell(s_{\varepsilon_1}) = \epsilon(\epsilon q^y)^\ell \Psi_\ell \left(q^{-\ell(y-1)} \left(\frac{1}{|W_{\varepsilon_1}|} m_{\varepsilon_1}^{(\ell)} + a \right) \right) \\
&= \epsilon^{\ell+1} q^\ell \left(\frac{1}{|W_{\varepsilon_1}|} \Psi_\ell \left(m_{\varepsilon_1}^{(\ell)} \right) + a \right) = \epsilon^{\ell+1} q^\ell \left(\frac{1}{|W_{\varepsilon_1}|} \Psi_1 \left(m_{\varepsilon_1}^{(1/\ell)} \right) + a \right) \\
&= \epsilon^\ell \Psi_1 \left(\epsilon q^\ell \left(\frac{m_{\varepsilon_1}^{(1/\ell)}}{|W_{\varepsilon_1}|} + a \right) \right) \\
&= \epsilon^\ell \Psi_1 \left(\epsilon q^\ell (a - b) + z \sum_{m=\max(\ell-r+1, 1)}^{\ell} (-1)^{\ell-m} q^{-(\ell-2m+1)} s_{(m, 1^{\ell-m+1-1})} \right. \\
&\quad \left. + z \sum_{m=\max(y-\ell+1, 1)}^{\ell-y+r} q^{-(\ell-2m+1)} (-1)^{m-\ell+y} s_{(m, 1^{m-\ell+y-1})} \right),
\end{aligned}$$

since $z = \epsilon q^y$. The result follows since $c = \epsilon q^\ell (a - b)$. \square

We would like to connect Corollary 4.4 to the Harish-Chandra images of the parameters $Z_V^{(\ell)}$ computed in Corollary 3.6. In order to do so, we will use the following result from [26, Lemma 3.5.1] (also see [7, Prop. 5.3]).

Theorem 4.5. *Let \mathfrak{g} be a finite-dimensional complex Lie algebra with a symmetric nondegenerate ad-invariant bilinear form and let $U = U_{\mathfrak{h}}\mathfrak{g}$ be the corresponding Drinfeld-Jimbo quantum group with R -matrix \mathcal{R} . Let ν be a dominant integral weight so that the irreducible module $L(\nu)$ of highest weight ν is finite-dimensional and let s_ν be the Weyl character of $L(\nu)$. Then*

$$(\text{id} \otimes \text{qtr}_{L(\nu)})(\mathcal{R}_{21}\mathcal{R}) \quad \text{acts on } L(\mu) \text{ by } \text{ev}_{2(\mu+\rho)}(s_\nu) \text{id}_{L(\mu)},$$

where $\text{ev}_\gamma: \mathbb{C}[\mathfrak{h}^*] \rightarrow \mathbb{C}$ are the algebra homomorphisms given by $\text{ev}_\gamma(e^\tau) = q^{(\gamma, \tau)}$ for $\gamma, \tau \in \mathfrak{h}^*$.

For $\mathfrak{g} = \mathfrak{so}_{2r+1}$, \mathfrak{sp}_{2r} or \mathfrak{so}_{2r} , the Turaev-Wenzl identity almost provides an inverse to the Harish-Chandra homomorphism. With $\varepsilon_1, \dots, \varepsilon_r$ as in (2.9), converting variable alphabets from

$$Y = \sum_{i \in \hat{V}} e^{\varepsilon_i} \quad \text{to} \quad X = \sum_{i \in \hat{V}} L_i^2, \quad \text{then} \quad \text{ev}_{2\lambda}(s_\mu(Y)) = \text{ev}_\lambda(s_\mu(X)).$$

Thus, Theorem 4.5 in combination with the Harish-Chandra isomorphism in Theorem 3.1 says that $\text{ev}_\mu(\pi_0(\Psi_1(s_\nu))) = \text{ev}_{2(\mu+\rho)} s_\nu(Y) = \text{ev}_{\mu+\rho} s_\nu(X) = \text{ev}_\mu(\sigma_\rho(s_\nu(X)))$. Hence

$$\pi_0(\Psi_1(s_\nu)) = \sigma_\rho(s_\nu(X)). \quad (4.11)$$

The modification rules of [17, §2.4] are used to convert the universal Weyl characters appearing in Corollary 3.6 to actual Weyl characters s_λ . In general, either $sp_\lambda(X) = 0$ or there is a unique dominant weight μ and a uniquely determined sign such that $sp_\lambda(X) = \pm s_\mu$, and similarly for the orthogonal cases. In particular, if $\ell(\lambda) < r$ then $sp_\lambda(X) = s_\lambda$ in the symplectic case and $so_\lambda(X) = s_\lambda$ in the orthogonal case [17, Prop. 2.2.1]. In view of (4.11), the conversion from universal Weyl characters to actual Weyl characters provides the equivalence between Corollary 4.4 and Corollary 3.6.

References

- [1] S. Ariki, A. Mathas and H. Rui, *Cyclotomic Nazarov-Wenzl algebras*, Nagoya Math. J. **182** (2006), 47–134, MR2235339, arXiv:math.QA/0506467.
- [2] P. Baumann, *On the center of quantized enveloping algebras*, J. Algebra **203** no. 1 (1998) 244–260. MR1620662
- [3] A. Beliakova and C. Blanchet, *Skein construction of idempotents in Birman-Murakami-Wenzl algebras*, Math. Ann. **321** no. 2 (2001), 347–373. MR 1866492
- [4] N. Bourbaki, *Groupes et algèbres de Lie*, Masson, Paris, 1990.
- [5] V. Chari, A. Pressley, *A guide to quantum groups*, Cambridge University Press, Cambridge, 1995. MR1358358
- [6] Z. Daugherty, A. Ram, R. Virk, *Affine and graded BMW algebras: The center*, arXiv:1105.4207
- [7] V. G. Drinfel'd, *On almost cocommutative Hopf algebras*, Leningrad Math. J. **1** (1990) 321–342. MR1025154
- [8] I. M. Gelfand, *Center of the infinitesimal group ring*, Mat. Sb., Nov. Ser. **26** (68) (1950) 103–112. Zbl. **35**:300.
- [9] F. Goodman, *Comparison of admissibility conditions for cyclotomic Birman-Wenzl-Murakami algebras*, J. Pure and Applied Algebra **214** (2010), 2009–2016. MR2645333 arXiv:0905.4258
- [10] F. Goodman, *Admissibility conditions for degenerate cyclotomic BMW algebras*, Comm. Algebra **39** (2011), no. 2, 452–461. MR2773313 arXiv:0905.4253
- [11] F. Goodman, P. de la Harpe and V.F.R. Jones, *Coxeter graphs and towers of algebras*, Mathematical Sciences Research Institute Publications **14**, Springer-Verlag, 1980. MR0999799
- [12] F. Goodman and H. Hauschild, *Affine Birman-Wenzl-Murakami algebras and tangles in the solid torus*, Fundamenta Mathematicae **190** (2006), 77–137. MR2232856 arXiv:math.QA/0411155
- [13] F. Goodman and H. Mosley, *Cyclotomic Birman-Wenzl-Murakami algebras I: Freeness and realization as tangle algebras*, J. Knot Theory and Its Ramifications, **18**, no. 8, (2009) 1089–1127. MR2554337 arXiv:math/0612064.
- [14] F. Goodman and H. Mosley, *Cyclotomic Birman-Wenzl-Murakami algebras II: Admissibility relations and freeness*, Algebr. Represent. Theory **14** (2011), no. 1, 139. MR2763289 arXiv:math.QA/0612065
- [15] R. Häring-Oldenburg, *The reduced Birman-Wenzl algebra of Coxeter type B*, J. Algebra **213** (1999), 437–466. MR1673464
- [16] T. Halverson and A. Ram, *Characters of algebras containing a Jones basic construction: The Temperley-Lieb, Brauer, Okada and Birman-Wenzl algebras*, Adv. Math. **116** (1995) 263–321. MR1363766

- [17] K. Koike and I. Terada, *Young diagrammatic methods for the representation theory of the classical groups of type B_n, C_n, D_n* , J. Algebra **107** (1987), 466-511. MR0885807
- [18] R. Leduc and A. Ram, *A ribbon Hopf algebra approach to the irreducible representations of centralizer algebras: The Brauer, Birman-Wenzl, and Type A Iwahori-Hecke algebras*, Advances Math. **125** (1997), 1-94. MR1427801
- [19] A. Molev, *Yangians and classical Lie algebras*, Mathematical Surveys and Monographs **143**, American Mathematical Society, Providence, RI, 2007. xviii+400 pp. ISBN: 978-0-8218-4374-1 MR2355506
- [20] A. Molev and N. Rozhkovskaya, *Characteristic maps for the Brauer algebra*, arXiv:1112.0620v1.
- [21] M. Nazarov, *Young's Orthogonal Form for Brauer's Centralizer Algebra*, J. Algebra **182** (1996), 664–693. MR1398116
- [22] R. Orellana and A. Ram, *Affine braids, Markov traces and the category \mathcal{O}* , in *Proceedings of the International Colloquium on Algebraic Groups and Homogeneous Spaces Mumbai 2004*, V.B. Mehta ed., Tata Institute of Fundamental Research, Narosa Publishing House, Amer. Math. Soc. (2007), 423–473. MR2348913 arXiv:math/0401317
- [23] A. M. Perelomov and V. S. Popov, *A generating function for Casimir operators*, Dokl. Akad. Nauk SSSR **174** (1967), 1021–1023 (Russian). MR0214703
- [24] A. M. Perelomov and V. S. Popov, *Casimir operators for the orthogonal and symplectic groups*, Jadernaja Fiz. **3** 1127–1134 (Russian); translated as Soviet J. Nuclear Phys. **3** (1996) 819–824. MR0205621
- [25] N. Yu. Reshetikhin, L.A. Takhtadzhyan and L.D. Faddeev, *Quantization of Lie groups and Lie algebras*, Algebra i Analiz **1** (1989) 178-206 and Leningrad Math. J. **1** (1990), 193-225. MR1015339
- [26] V. Turaev and H. Wenzl, *Quantum invariants of 3-manifolds associated with classical simple Lie algebras*, Int. J. of Math. **4**, No. 2 (1993), 323–358. MR1217386
- [27] H. Weyl, *Classical groups, their invariants and representations*, Princeton Univ. Press, Princeton NJ, 1946. MR0000255
- [28] S. Wilcox and S. Yu, *The cyclotomic BMW algebra associated with the two string type B braid group*, to appear in Communications in Algebra, arXiv:math.RT/0611518.
- [29] S. Wilcox and S. Yu, *On the freeness of the cyclotomic BMW algebras: admissibility and an isomorphism with the cyclotomic Kauffman tangle algebras*, arXiv:0911.5284.
- [30] S. Yu, *The cyclotomic Birman-Murakami-Wenzl algebras*, Ph.D. Thesis, University of Sydney (2007). arXiv:0810.0069.
- [31] D. P. Zhelobenko, *Compact Lie groups and their representations*, Translated from the Russian by Israel Program for Scientific Translations. Translations of Mathematical Monographs, Vol. 40. American Mathematical Society, Providence, RI (1973). MR0473097