A REMARK ON EHRESMANN'S FIBRATION THEOREM

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If $f\colon Z \to Y$ is a smooth proper morphism of smooth varieties, and $\mathcal{L}$ a local system on Z, then the sheaves $R^q f_*\mathcal{L}$ are local systems on Y. This is typically seen as a consequence of Ehresmann's Theorem - f is a topological fiber bundle over each component of Y ([Vo, Theorem 9.3] is a convenient reference). This note records that the cohomological consequence holds without the smooth assumption on Y or Z.

Conventions. A 'sheaf' means a 'sheaf of vector spaces over some fixed field', and 'variety' = 'separated reduced scheme of finite type over Spec$(\mathbf{C})$'. Sheaves on varieties are with respect to the complex analytic site. A proper map of topological spaces is a separated and universally closed map.

J-L. Verdier asserts the following without the locally connected hypothesis [Ve, Lemme 2.2.2]. I was unable to understand his proof without this assumption.

1. **Lemma.** *Let $p\colon X \to Y$ be a proper surjective map of topological spaces. Assume X is locally connected. Let $\mathcal{F}$ be a sheaf on Y with finite dimensional stalks. If $p^*\mathcal{F}$ is a local system, then so is $\mathcal{F}$.*

Proof. Let $y \in Y$. The stalk $\mathcal{F}_y$ is finite dimensional, so there exist sections $s_1, \dots, s_n$, of $\mathcal{F}$ over some open neighborhood of y, which restrict to a basis of $\mathcal{F}_y$. Since our problem is local, we may assume this neighborhood is all of Y. Let $\mathcal{G}$ be the constant sheaf on Y with stalk span$\{s_1, \dots, s_n\}$. Then the evident map $u\colon \mathcal{G} \to \mathcal{F}$ induces an isomorphism $\mathcal{G}_y \xrightarrow{\sim} \mathcal{F}_y$. Consequently, p^*u induces isomorphisms:

$$(p^*\mathcal{G})_x \xrightarrow{\sim} (p^*\mathcal{F})_x \quad \text{for all } x \in p^{-1}(y).$$

For a locally connected space, the set of points at which a morphism of local systems induces an isomorphism on stalks defines an open set. Hence, the set $V \subset X$ of points at which p^*u induces isomorphisms on stalks is open. As p is proper, $U = Y - f(X - V)$ is an open neighborhood of y. As p is surjective, u yields an isomorphism $\mathcal{G}|_U \xrightarrow{\sim} \mathcal{F}|_U$. □

2. **Proposition.** *Let $f\colon Z \to Y$ be a smooth and proper morphism of varieties. Let $\mathcal{L}$ be a local system on Z with finite dimensional stalks. Then the sheaves $R^q f_*\mathcal{L}$ are local systems.*

Proof. Resolution of singularities (the version in [BP] suffices), the Lemma and proper base change reduce us to the situation where Z and Y are smooth. Here the usual form of Ehresmann's Theorem applies. □

References

[BP] F. Bogomolov, T. Pantev, *Weak Hironaka Theorem*, arXiv:alg-geom/9603019v2.

[Ve] J-L. Verdier, *Classe d'Homologie associée un Cycle*, Asterisque **36-37**, p. 101-151 (1976).

[Vo] C. Voisin, *Hodge Theory and Complex Algebraic Geometry I*, Cambridge Studies in Math. **76** (2002).