FOURIER TRANSFORM AND EXPONENTIAL HODGE MODULES

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1. Introduction.

1.1. The purpose of this note is to prove invertibility of Fourier transform for exponential Hodge modules (Theorem 5.3). The arguments transfer to motivic, ℓ-adic, etc. setups with minor notational changes. The results are probably well known to experts. No claims to originality are being made.

This is primarily a sanity check for my attempts to understand N. Katz's ideas on exponential sums [K]. Informally, the objects of the categories $E(X)$ defined below are classes of "classical" objects of $D(X \times \mathbb{G}_a)$. Given a classical object $K \in D(X \times \mathbb{G}_a)$, its class in $E(X)$ represents the *classical* object $\pi_{X!}(K \otimes \mu^* e^t) \in D(X)$. Here $\pi_X : X \times \mathbb{G}_a \to X$ and $\mu : X \times \mathbb{G}_a \to \mathbb{G}_a$ are the projections, and e^t is the "standard exponential object" in $D(\mathbb{G}_a)$: the D-module given by the differential equation $f'(t) - f(t) = 0$ in the de Rham setting, or an Artin-Schreier sheaf in the ℓ-adic setting over finite fields (these give classical exponential sums via the Grothendieck trace formula). The above formula gives a realization functor $E(X) \to D(X)$ whenever such an exponential sheaf exists. The categories $E(X)$ are equipped with a natural "exponential kernel" (the object **E** in §4) that *always* exists in *any* reasonable sheaf formalism. The point is that it exists even when a classical canonical exponential sheaf doesn't: Artin-Schreier sheaves are char p objects and there are $p-1$ of them over the prime field $\mathbb{F}_p$ with no reasonable way to choose amongst them compatibly across p; the exponential D-module is irregular and does not underlie a mixed Hodge module; no such local system on $\mathbb{A}^1(\mathbb{C})$ exists in the classical topology.

The constructions below can be made essentially verbatim for ℓ-adic sheaves on schemes over $\mathbb{Z}[1/\ell]$, say. The objects of $E(X)$ and the Fourier transform on $E(X)$ are then "universal" versions of the classical notions, specializing to them under the evident realizations. The $E(X)$ are thus vehicles for cross-characteristic comparisons: a direct comparison via Riemann-Hilbert is generally not possible because the relevant ℓ-adic sheaves/D-modules usually have wild ramification/irregular singularities.

2. Conventions.

2.1. 'Map' and 'morphism' will be used interchangeably. 'Canonical map' will be used as a synonym for 'natural transformation of functors'.

2.2. For a variety X (i.e., a separated reduced scheme of finite type over $\mathrm{Spec}(\mathbb{C})$), we write $MHM(X)$ for the abelian category of mixed Hodge modules on X, and $D(X)$ for its bounded derived category (see [S]). Functors on derived categories will always be derived. I.e., we write f_* instead of Rf_*, etc.

2.3. The r-th Tate twist will be denoted (r). The constant sheaf in $D(X)$ (i.e., the monoidal unit for $\otimes$) will be denoted $\underline{\mathbb{Q}}^H_X$.

3. Exponential objects.

3.1. Let X be a variety. Write:

$$\pi_X : X \times \mathbb{G}_a \to X$$

for the projection map. Then $\pi_X^*[1]$ is t-exact and the essential image of

$$\pi_X^*[1] \colon MHM(X) \to MHM(X \times \mathbb{G}_a)$$

is a Serre subcategory of $MHM(X \times \mathbb{G}_a)$ (see [BBD, Corollaire 4.2.6.2]). Hence,

$$E(X) = D(X \times \mathbb{G}_a)/\pi_X^* D(X)$$

is a triangulated category. That is, $E(X)$ is the category obtained from $D(X \times \mathbb{G}_a)$ by inverting all morphisms whose cone is isomorphic to an object of the form $\pi_X^* K$, with $K \in D(X)$. Informally, $E(X)$ is obtained from $D(X \times \mathbb{G}_a)$ by killing all objects that come from $D(X)$.

3.2. Given a morphism $f : X \to Y$, the functor $(f \times \mathrm{id}_{\mathbb{G}_a})^*$ induces a functor:

$$\boldsymbol{f}^* : E(Y) \to E(X).$$

Proper base change implies that $(f \times \mathrm{id}_{\mathbb{G}_a})_!$ induces a functor:

$$\boldsymbol{f}_! : E(X) \to E(Y).$$

3.3. Using $(f \times \mathrm{id}_{\mathbb{G}_a})_*$ and $(f \times \mathrm{id}_{\mathbb{G}_a})^!$ one also obtains functors $\boldsymbol{f}_*$ and $\boldsymbol{f}^!$ on the categories $E(X)$, since π_X is smooth. Similarly, Verdier duality is available. However, we will make no use of these (but see the remark in §5.4).

3.4. To avoid confusion, the convention used throughout this document is that $*$/!-pullback/pushforward functors in boldface are on $E(X)$, functors in ordinary font are the usual such functors on $D(X \times \mathbb{G}_a)$, $D(X)$, etc.

3.5. Write

$$\mathrm{add} \colon \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a$$

for the group operation. Consider the diagram:

$$\begin{array}{ccccc} & & X \times \mathbb{G}_a \times \mathbb{G}_a & \xrightarrow{\mathrm{id}_X \times \mathrm{add}} & X \times \mathbb{G}_a \\ & \swarrow^{p_{12}} & & \searrow^{p_{13}} & \\ X \times \mathbb{G}_a & & & & X \times \mathbb{G}_a \end{array}$$

where p_{12} and p_{13} are projections to the indicated factors. For $K, L \in D(X \times \mathbb{G}_a)$, set:

$$K \overset{+}{\otimes} L = (\mathrm{id}_X \times \mathrm{add})_!(p_{12}^* K \otimes p_{13}^* L).$$

This defines a symmetric monoidal structure on $D(X \times \mathbb{G}_a)$ with unit object given by

$$\mathbf{1}_X = (\mathrm{id}_X \times i_0)_! \underline{\mathbb{Q}}_X^H,$$

where $i_0 \colon \mathrm{Spec}(\mathbb{C}) \to \mathbb{G}_a$ is the unit for the group operation. Proper base change yields that this induces a symmetric monoidal structure on $E(X)$.

3.6. "Classical" proper base change implies that $f^* \colon E(Y) \to E(X)$ is $\overset{+}{\otimes}$-monoidal and that $f_!$ satisfies proper base change. The projection formula with respect to $\overset{+}{\otimes}$ also holds in $E(X)$: this uses both classical proper base change and the classical projection formula. The standard tensor product $\otimes$ does *not* descend to $E(X)$.

3.7. The quotient property of $E(X)$ has not been used so far. It will enter in a crucial way in Proposition 4.4. Regardless, note that we could have defined convolution using add_* instead of $\mathrm{add}_!$. Although this $*$-convolution differs from our !-convolution on $D(X \times \mathbb{G}_a)$, they become canonically isomorphic in $E(X)$.[1]

4. The exponential kernel.

4.1. Let $\Delta \colon \mathbb{G}_a \to \mathbb{G}_a \times \mathbb{G}_a$ be the diagonal morphism. Define $\mathbf{E} \in E(\mathbb{G}_a)$ by:

$$\mathbf{E} = \Delta_! \underline{\mathbb{Q}}_{\mathbb{G}_a}^H.$$

Given morphisms $f, g \colon X \to \mathbb{G}_a$, set $f + g = \mathrm{add} \circ (f \times g)$.

4.2. Proposition (Additivity)**.** *We have:*

$$f^* \mathbf{E} \overset{+}{\otimes} g^* \mathbf{E} \simeq (f + g)^* \mathbf{E}.$$

Proof. Indeed, let $\gamma_g \colon X \to X \times \mathbb{G}_a$ be the graph $\gamma_g(x) = (x, g(x))$ and let

$$\tilde{\gamma}_f \colon X \times \mathbb{G}_a \to X \times \mathbb{G}_a \times \mathbb{G}_a$$

be given by $\tilde{\gamma}_f(x, t) = (x, f(x), t)$. Then by "classical" proper base change:

$$f^* \Delta_! \underline{\mathbb{Q}}_{\mathbb{G}_a}^H \overset{+}{\otimes} g^* \Delta_! \underline{\mathbb{Q}}_{\mathbb{G}_a}^H \simeq (\mathrm{id}_X \times \mathrm{add})_! (\tilde{\gamma}_{f!} \underline{\mathbb{Q}}_{X \times \mathbb{G}_a}^H \otimes p_{13}^* \gamma_{g!} \underline{\mathbb{Q}}_X^H).$$

[1] The cone of the forget supports map from !-convolution to $*$-convolution comes from $D(X)$. See [KS, §4.4, Lemma 1] for an argument in the case $X = \mathrm{Spec}(\mathbb{C})$.

By the "classical" projection formula this is isomorphic to:

$$(\mathrm{id}_X \times \mathrm{add})_! \tilde{\gamma}_{f!} \tilde{\gamma}_f^* p_{13}^* \gamma_{g!} \underline{\mathbb{Q}}_X^H.$$

But $p_{13} \circ \tilde{\gamma}_f = \mathrm{id}$, so this becomes:

$$(\mathrm{id}_X \times \mathrm{add})_! \tilde{\gamma}_{f!} \gamma_{g!} \underline{\mathbb{Q}}_X^H.$$

Now $(\mathrm{id}_X \times \mathrm{add}) \circ \tilde{\gamma}_f \circ \gamma_g$ is the graph of $f+g$. This gives us the desired result. □

4.3. Let $\pi\colon V \to S$ be a vector bundle of constant rank $r \geq 1$. Let $V^\vee \to S$ be the dual bundle, and $m\colon V \times_S V^\vee \to \mathbb{G}_a$ the canonical pairing. Write $q\colon V \times_S V^\vee \to V^\vee$ for the projection. The following result is the crucial ingredient for the eventual invertibility of the Fourier transform. The quotient property of $E(X)$ is essential: the statement fails in $D(V^\vee \times \mathbb{G}_a)$.

4.4. Proposition (Orthogonality). *In $E(V^\vee)$, we have:*

$$\boldsymbol{q}_! \boldsymbol{m}^* \mathbf{E} \simeq \boldsymbol{s}_! \mathbf{1}_S[-2r](-r),$$

where $s\colon S \to V^\vee$ is the zero section.

Proof. Consider the object

$$K = (q \times \mathrm{id}_{\mathbb{G}_a})_! (m \times \mathrm{id}_{\mathbb{G}_a})^* \Delta_! \underline{\mathbb{Q}}_{\mathbb{G}_a}^H$$

in $D(V^\vee \times \mathbb{G}_a)$, and the distinguished triangle:

$$j_! j^* K \to K \to i_* i^* K \xrightarrow{[1]}$$

where $i = s \times \mathrm{id}_{\mathbb{G}_a}$ and j is the complementary open immersion. We will show that

$$j^* K \simeq \underline{\mathbb{Q}}_{(V^\vee \setminus S) \times \mathbb{G}_a}^H [-2(r-1)](-(r-1))$$

in $D((V^\vee \setminus S) \times \mathbb{G}_a)$, and that

$$i^* K \simeq (\mathrm{id}_S \times i_0)_! \underline{\mathbb{Q}}_S^H [-2r](-r),$$

where $i_0\colon \mathrm{Spec}(\mathbb{C}) \to \mathbb{G}_a$ is the group unit. This will give us the result, as the class of

$$j_! \underline{\mathbb{Q}}_{(V^\vee \setminus S) \times \mathbb{G}_a}^H = 0$$

in $E(V^\vee) = D(V^\vee \times \mathbb{G}_a)/\pi_{V^\vee}^* D(V^\vee)$.

For j^*K, we have a cartesian square:

$$\begin{array}{ccc}
V \times_S (V^\vee \setminus S) \times \mathbb{G}_a & \xrightarrow{\tilde{j}} & V \times_S V^\vee \times \mathbb{G}_a \\
\big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle q \times \mathrm{id}_{\mathbb{G}_a}} \\
(V^\vee \setminus S) \times \mathbb{G}_a & \xrightarrow{j} & V^\vee \times \mathbb{G}_a
\end{array}$$

where $\tilde{j}$ is the evident inclusion and $p(x,y,z) = (y,z)$. So by proper base change:

$$j^*K \simeq p_!((m \times \mathrm{id}_{\mathbb{G}_a}) \circ \tilde{j})^* \Delta_! \underline{\mathbb{Q}}^H_{\mathbb{G}_a}.$$

Now apply proper base change again with the cartesian squares:

$$\begin{array}{ccccc} V \times_S (V^\vee \setminus S) & \xrightarrow{j'} & V \times_S V^\vee & \xrightarrow{m} & \mathbb{G}_a \\ \downarrow{\scriptstyle \tilde{\gamma}_m} & & {\scriptstyle \gamma_m}\downarrow & & \downarrow{\scriptstyle \Delta} \\ V \times_S (V^\vee \setminus S) \times \mathbb{G}_a & \xrightarrow{\tilde{j}} & V \times_S V^\vee \times \mathbb{G}_a & \xrightarrow{m \times \mathrm{id}_{\mathbb{G}_a}} & \mathbb{G}_a \times \mathbb{G}_a \end{array}$$

where γ_m is the graph of m, the map $\tilde{\gamma}_m$ is its restriction to $V \times_S (V^\vee \setminus S)$, and j' is the evident inclusion. This gives

$$j^*K \simeq p_! \tilde{\gamma}_{m!} \underline{\mathbb{Q}}^H_{V \times_S (V^\vee \setminus S)}.$$

The map

$$p \circ \tilde{\gamma}_m \colon V \times_S (V^\vee \setminus S) \to (V^\vee \setminus S) \times \mathbb{G}_a$$

is given by $(x,y) \mapsto (y, m(x,y))$. This is a Zariski locally trivial $\mathbb{A}^{r-1}$-bundle.[2] Hence,

$$p_! \tilde{\gamma}_{m!} \underline{\mathbb{Q}}^H_{V \times_S (V^\vee \setminus S)} \simeq \underline{\mathbb{Q}}^H_{(V^\vee \setminus S) \times \mathbb{G}_a}[-2(r-1)](-(r-1)).$$

For i^*K, we have a cartesian square:

$$\begin{array}{ccc} V \times \mathbb{G}_a & \xrightarrow{\tilde{i}} & V \times_S V^\vee \times \mathbb{G}_a \\ {\scriptstyle \pi \times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle q \times \mathrm{id}_{\mathbb{G}_a}} \\ S \times \mathbb{G}_a & \xrightarrow{i} & V^\vee \times \mathbb{G}_a \end{array}$$

where $\tilde{i}(x,z) = (x, s\pi(x), z)$. Hence, by proper base change:

$$i^*K \simeq (\pi \times \mathrm{id}_{\mathbb{G}_a})_!((m \times \mathrm{id}_{\mathbb{G}_a}) \circ \tilde{i})^* \Delta_! \underline{\mathbb{Q}}^H_{\mathbb{G}_a}.$$

Now apply proper base change again with the cartesian squares:

$$\begin{array}{ccccc} V & \xrightarrow{i'} & V \times_S V^\vee & \xrightarrow{m} & \mathbb{G}_a \\ {\scriptstyle \mathrm{id}_V \times i_0}\downarrow & & {\scriptstyle \gamma_m}\downarrow & & \downarrow{\scriptstyle \Delta} \\ V \times \mathbb{G}_a & \xrightarrow{\tilde{i}} & V \times_S V^\vee \times \mathbb{G}_a & \xrightarrow{m \times \mathrm{id}_{\mathbb{G}_a}} & \mathbb{G}_a \times \mathbb{G}_a \end{array}$$

where $i'(x) = (x, s\pi(x))$, and $i_0 \colon \mathrm{Spec}(\mathbb{C}) \to \mathbb{G}_a$ is the group unit. This gives

$$(\pi \times \mathrm{id}_{\mathbb{G}_a})_!((m \times \mathrm{id}_{\mathbb{G}_a}) \circ \tilde{i})^* \Delta_! \underline{\mathbb{Q}}^H_{\mathbb{G}_a} \simeq ((\pi \times \mathrm{id}_{\mathbb{G}_a}) \circ (\mathrm{id}_V \times i_0))_! \underline{\mathbb{Q}}^H_V.$$

But $\pi \colon V \to S$ is a rank r vector bundle, so

$$i^*K \simeq (\mathrm{id}_S \times i_0)_! \underline{\mathbb{Q}}^H_S[-2r](-r). \qquad \square$$

[2]The heart of the matter is that the kernel of a non-zero linear form on a vector space is a hyperplane.

5. Fourier transform.

5.1. Let $\pi\colon V \to S$ be a vector bundle of constant rank $r \geq 1$. Write $V^\vee \to S$ for the dual bundle, and

$$m\colon V \times_S V^\vee \to \mathbb{G}_a$$

for the canonical pairing. So we have a diagram:

$$\begin{array}{ccccc} & & V \times_S V^\vee & \xrightarrow{\;m\;} & \mathbb{G}_a \\ & \swarrow^{p} & & \searrow^{q} & \\ V & & & & V^\vee \end{array}$$

where p and q are the evident projections.

5.2. Define $\mathbf{FT}\colon E(V) \to E(V^\vee)$ by:

$$\mathbf{FT}_V(K) = \boldsymbol{q}_!(\boldsymbol{p}^*K \overset{+}{\otimes} \boldsymbol{m}^*\mathbf{E})[r].$$

5.3. Theorem. *Let $a\colon V \xrightarrow{\sim} V^{\vee\vee}$ be the isomorphism defined by $a(v) = -m(v,-)$. Then we have a canonical isomorphism:*

$$\mathbf{FT}_{V^\vee} \circ \mathbf{FT}_V(K) \simeq \boldsymbol{a}_!K(-r).$$

Proof. Let p_1, p_2, p_{12}, etc., denote the projections from $V \times_S V^\vee \times_S V^{\vee\vee}$ to the named factors. We also have projections:

$$\begin{array}{ccccccccc} & V \times_S V^\vee & & & V^\vee \times_S V^{\vee\vee} & & & V \times_S V^{\vee\vee} & \\ \swarrow^{p} & & \searrow^{q} & \swarrow^{\tilde{p}} & & \searrow^{\tilde{q}} & \swarrow^{\bar{q}} & & \searrow^{\bar{p}} \\ V & & & V^\vee & & & V^{\vee\vee} & & V \end{array}$$

and canonical pairings:

$$m\colon V \times_S V^\vee \to \mathbb{G}_a, \qquad \tilde{m}\colon V^\vee \times_S V^{\vee\vee} \to \mathbb{G}_a.$$

These fit into cartesian squares:

$$\begin{array}{ccc} V \times_S V^\vee \times_S V^{\vee\vee} & \xrightarrow{\;\alpha\;} & V^\vee \times_S V^{\vee\vee} \\ \downarrow{\scriptstyle p_{13}} & & \downarrow{\scriptstyle \tilde{q}} \\ V \times_S V^{\vee\vee} & \xrightarrow{\;\beta\;} & V^{\vee\vee} \end{array} \qquad \begin{array}{ccccc} V & \xrightarrow{\;\Delta\;} & V \times_S V & \xrightarrow{\;\mathrm{id}_V \times a\;} & V \times_S V^{\vee\vee} \\ \downarrow{\scriptstyle \pi} & & & & \downarrow{\scriptstyle \beta} \\ S & & \xrightarrow[\;s\;]{} & & V^{\vee\vee} \end{array}$$

$$\begin{array}{ccc} V \times_S V^\vee \times_S V^{\vee\vee} & \xrightarrow{\;p_{12}\;} & V \times_S V^\vee \\ \downarrow{\scriptstyle p_{23}} & & \downarrow{\scriptstyle q} \\ V^\vee \times_S V^{\vee\vee} & \xrightarrow{\;\tilde{p}\;} & V^\vee \end{array}$$

where Δ is the diagonal, $s\colon S\to V^{\vee\vee}$ is the zero section, $\beta(x,z)=z-a(x)$, and $\alpha(x,y,z)=(y,z-a(x))$. Now:

$$\begin{aligned}
\mathbf{FT}_{V^\vee}\circ\mathbf{FT}_V(K) &= \widetilde{q}_!(\widetilde{p}^*q_!(p^*K\overset{+}{\otimes}m^*\mathbf{E})\overset{+}{\otimes}\widetilde{m}^*\mathbf{E})[2r] && \text{(by definition)}\\
&\simeq \widetilde{q}_!(p_{23!}p_{12}^*(p^*K\overset{+}{\otimes}m^*\mathbf{E})\overset{+}{\otimes}\widetilde{m}^*\mathbf{E})[2r] && \text{(proper base change)}\\
&\simeq \widetilde{q}_!p_{23!}(p_{12}^*(p^*K\overset{+}{\otimes}m^*\mathbf{E})\overset{+}{\otimes}p_{23}^*\widetilde{m}^*\mathbf{E})[2r] && \text{(projection formula)}\\
&\simeq \bar{q}_!p_{13!}(p_{12}^*p^*K\overset{+}{\otimes}p_{12}^*m^*\mathbf{E}\overset{+}{\otimes}p_{23}^*\widetilde{m}^*\mathbf{E})[2r] && (\widetilde{q}\circ p_{23}=\bar{q}\circ p_{13})\\
&\simeq \bar{q}_!p_{13!}(p_{13}^*\bar{p}^*K\overset{+}{\otimes}p_{12}^*m^*\mathbf{E}\overset{+}{\otimes}p_{23}^*\widetilde{m}^*\mathbf{E})[2r] && (p\circ p_{12}=\bar{p}\circ p_{13})\\
&\simeq \bar{q}_!p_{13!}(p_{13}^*\bar{p}^*K\overset{+}{\otimes}\alpha^*\widetilde{m}^*\mathbf{E})[2r] && \text{(additivity)}\\
&\simeq \bar{q}_!(\bar{p}^*K\overset{+}{\otimes}p_{13!}\alpha^*\widetilde{m}^*\mathbf{E})[2r] && \text{(projection formula)}\\
&\simeq \bar{q}_!(\bar{p}^*K\overset{+}{\otimes}\beta^*\widetilde{q}_!\widetilde{m}^*\mathbf{E})[2r] && \text{(proper base change)}\\
&\simeq \bar{q}_!(\bar{p}^*K\overset{+}{\otimes}\beta^*s_!\mathbf{1}_S)(-r) && \text{(orthogonality)}\\
&\simeq \bar{q}_!(\bar{p}^*K\overset{+}{\otimes}(\mathrm{id}_V\times a)_!\Delta_!\mathbf{1}_V)(-r) && \text{(proper base change)}\\
&\simeq \bar{q}_!(\mathrm{id}_V\times a)_!\Delta_!(\Delta^*(\mathrm{id}_V\times a)^*\bar{p}^*K\overset{+}{\otimes}\mathbf{1}_V)(-r) && \text{(projection formula)}\\
&\simeq \bar{q}_!(\mathrm{id}_V\times a)_!\Delta_!(K\overset{+}{\otimes}\mathbf{1}_V)(-r) && (\bar{p}\circ(\mathrm{id}_V\times a)\circ\Delta=\mathrm{id})\\
&\simeq a_!K(-r) && (\bar{q}\circ(\mathrm{id}_V\times a)\circ\Delta=a).
\end{aligned}$$

□

5.4. Adjoints to a functor are unique up to canonical isomorphism. So the above invertibility implies, formally, the Fourier miracle: $\mathbf{FT}_V$ commutes with Verdier duality (up to Tate twist). The argument is due to J.-L. Verdier (see [L, Théoréme 4.1]). This has immediate consequences for t-exactness and purity (in the sense of weights). However, we have discussed neither t-structures nor weight structures on $E(X)$.

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