1. Conventions.

1.1. Work over a ring R that is assumed to be nice enough so that we have at our disposal an ℓ-adic formalism of the six Grothendieck operations on the category of separated schemes of finite type over Spec(R). We write $D(X)$ instead of $D^b_c(X, \bar{\mathbf{Q}}_\ell)$. Further, all functors are implicitly derived. I.e., we write $f_!$ instead of $Rf_!$, etc.

1.2. ‘Map’ and ‘morphism’ will be used interchangably. ‘Canonical map’ will be used as a synonym for ‘natural transformation of functors’.

2. Overworld.

2.1. Define:

$$E(X) = D(X \times \mathbf{A}^1)/p^* D(X),$$

where $p_1 : X \times \mathbf{A}^1 \to X$ is the projection. I.e., $E(X)$ is the Verdier localization of $D(X \times \mathbf{A}^1)$ at the thick subcategory consisting of objects isomorphic to $p_1^* K$, for $K \in D(X)$. In other words, morphisms in $D(X \times \mathbf{A}^1)$ whose cone isomorphic to $p_1^* K$, for $K \in D(X)$, have been inverted.

2.2. Define $\mathbb{1}_X \in E(X)$ via:

$$\mathbb{1}_X = \delta_{X!} \bar{\mathbf{Q}}_\ell,$$

where $\delta_X : X \to X \times \mathbf{A}^1$ is the zero section.

2.3. Given $K, L \in E(X)$, define:

$$K \overset{+}{\otimes} L = (\mathrm{id} \times \sigma)_! (p_{12}^* K \otimes p_{13}^* L),$$

where $p_{12}, p_{13} : X \times \mathbf{A}^1 \times \mathbf{A}^1 \to X \times \mathbf{A}^1$ are the projections to the indicated factors, and $\sigma : \mathbf{A}^1 \times \mathbf{A}^1 \to \mathbf{A}^1$ is the sum map $(s, t) \mapsto (s + t)$. This yields a symmetric monoidal structure on $E(X)$ with $\mathbb{1}_X$ as the monoidal unit.

2.4. Define $\mathbb{E} \in E(\mathbf{A}^1)$ via:

$$\mathbb{E} = \Delta_! \bar{\mathbf{Q}}_\ell,$$

where $\Delta : \mathbf{A}^1 \to \mathbf{A}^1 \times \mathbf{A}^1$ is the diagonal.

2.5. Given a morphism $f : X \to Y$, define $f^* : E(Y) \to E(X)$ and $f_! : E(X) \to E(Y)$ via $(f \times \mathrm{id})^*$ and $(f \times \mathrm{id})_!$ on $D(Y)$ and $D(X)$, respectively. That $f_! : E(X) \to E(Y)$ is well defined follows from proper base change.

2.6. “Classical” proper base change implies that f^* is monoidal with respect to $\overset{+}{\otimes}$. It also yields proper base change for functors on $E(X)$. Similarly, the “classical” projection formula and proper base change imply the projection formula (with $\overset{+}{\otimes}$) for functors on $E(X)$.

2.7. *(Additivity)*. Given morphisms $f, g: X \to \mathbf{A}^1$, we have:

$$f^*\mathbb{E} \overset{+}{\otimes} g^*\mathbb{E} \simeq (f+g)^*\mathbb{E}.$$

Indeed, let $\gamma_g: X \to X \times \mathbf{A}^1$ be the graph $\gamma_g(x) = (x, g(x))$ and let $\tilde{\gamma}_f: X \times \mathbf{A}^1 \to X \times \mathbf{A}^1 \times \mathbf{A}^1$ be given by $\tilde{\gamma}_f(x,t) = (x, f(x), t)$. Then by "classical" proper base change:

$$f^*\mathbb{E} \overset{+}{\otimes} g^*\mathbb{E} \simeq (\mathrm{id} \times \sigma)_!(\tilde{\gamma}_{f!}\bar{\mathbf{Q}}_\ell \otimes p_{13}^*\gamma_{g!}\bar{\mathbf{Q}}_\ell).$$

By the "classical" projection formula this is isomorphic to:

$$(\mathrm{id} \times \sigma)_!\tilde{\gamma}_{f!}\tilde{\gamma}_f^* p_{13}^*\gamma_{g!}\bar{\mathbf{Q}}_\ell.$$

But $p_{13} \circ \tilde{\gamma}_f = \mathrm{id}$, so this becomes:

$$(\mathrm{id} \times \sigma)_!\tilde{\gamma}_{f!}\gamma_{g!}\bar{\mathbf{Q}}_\ell.$$

Now $(\mathrm{id} \times \sigma) \circ \tilde{\gamma}_f \circ \gamma_g$ is the graph of $f+g$, i.e., the map $x \mapsto (x, f(x)+g(x))$. So another application of proper base change gives us the desired result.

2.8. *(Orthogonality)*. Let $\mu: \mathbf{A}^1 \times \mathbf{A}^1 \to \mathbf{A}^1$ be multiplication: $(x,y) \mapsto xy$, and write $p_2: \mathbf{A}^1 \times \mathbf{A}^1 \to \mathbf{A}^1$ for the second projection. Let $\delta: \mathrm{Spec}(R) \to \mathbf{A}^1$ be the zero section. Then

$$p_{2!}\mu^*\mathbb{E} \simeq \delta_!\mathbb{1}_{\mathrm{Spec}(R)}[-2](-1).$$

This is mostly bookkeeping, but it is also the first place the quotient property of $E(X)$ is needed. So let's go through the details by brute force. Consider the object $K = (p_2 \times \mathrm{id})_!(\mu \times \mathrm{id})^*\Delta_!\bar{\mathbf{Q}}_\ell$ in $D(\mathbf{A}^1 \times \mathbf{A}^1)$ and the distinguished triangle:

$$j_!j^*K \to K \to i_*i^*K \xrightarrow{[1]}$$

where $j: \mathbf{G}_m \times \mathbf{A}^1 \to \mathbf{A}^1 \times \mathbf{A}^1$ is the inclusion, and $i: \mathrm{Spec}(R) \times \mathbf{A}^1 \to \mathbf{A}^1 \times \mathbf{A}^1$ is the complement. We will show that $j^*K \simeq \bar{\mathbf{Q}}_\ell$ in $D(\mathbf{G}_m \times \mathbf{A}^1)$, and that $i^*K \simeq \delta_!\bar{\mathbf{Q}}_\ell[-2](-1)$. This will give us the result, since $j_!\bar{\mathbf{Q}}_\ell = 0$ in $E(\mathbf{A}^1) = D(\mathbf{A}^1 \times \mathbf{A}^1)/p_1^*D(\mathbf{A}^1)$.

For j^*K, we have a cartesian square:

$$\begin{array}{ccc}
\mathbf{G}_m \times \mathbf{A}^1 \times \mathbf{A}^1 & \xrightarrow{\tilde{j}} & \mathbf{A}^1 \times \mathbf{A}^1 \times \mathbf{A}^1 \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p_2 \times \mathrm{id}} \\
\mathbf{G}_m \times \mathbf{A}^1 & \xrightarrow{j} & \mathbf{A}^1 \times \mathbf{A}^1
\end{array}$$

where $\tilde{j}(x,y,z) = (y,x,z)$ and $\pi(x,y,z) = (x,z)$. So by proper base change:

$$j^*K \simeq \pi_!((\mu \times \mathrm{id}) \circ \tilde{j})^*\Delta_!\bar{\mathbf{Q}}_\ell.$$

Now apply proper base change again with the cartesian squares:

$$\begin{array}{ccccc}
\mathbf{G}_m \times \mathbf{A}^1 & \xrightarrow{j'} & \mathbf{A}^1 \times \mathbf{A}^1 & \xrightarrow{\mu} & \mathbf{A}^1 \\
\downarrow{\scriptstyle \tilde{\gamma}_\mu} & & \downarrow{\scriptstyle \gamma_\mu} & & \downarrow{\scriptstyle \Delta} \\
\mathbf{G}_m \times \mathbf{A}^1 \times \mathbf{A}^1 & \xrightarrow{\tilde{j}} & \mathbf{A}^1 \times \mathbf{A}^1 \times \mathbf{A}^1 & \xrightarrow{\mu \times \mathrm{id}} & \mathbf{A}^1 \times \mathbf{A}^1
\end{array}$$

where γ_μ is the graph of μ, $\tilde{\gamma}_\mu$ its restriction to $\mathbf{G}_m \times \mathbf{A}^1$, and $j'(x,y) = (y,x)$. This gives

$$j^*K \simeq \pi_!((\mu \times \mathrm{id}) \circ \tilde{j})^*\Delta_!\bar{\mathbf{Q}}_\ell \simeq \pi_!\tilde{\gamma}_{\mu!}\bar{\mathbf{Q}}_\ell.$$

But now $\pi\tilde{\gamma}_\mu(x,y) = (x, xy)$ - an isomorphism. Hence, $j^*K \simeq \bar{\mathbf{Q}}_\ell$.

For i^*K, we have a cartesian square:

$$\begin{array}{ccc} \mathbf{A}^1 \times \mathbf{A}^1 & \xrightarrow{\tilde{i}} & \mathbf{A}^1 \times \mathbf{A}^1 \times \mathbf{A}^1 \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_2 \times \mathrm{id}} \\ \mathbf{A}^1 & \xrightarrow{i} & \mathbf{A}^1 \times \mathbf{A}^1 \end{array}$$

where $\tilde{i}(x,y) = (x,0,y)$. Hence, by proper base change:

$$i^*K \simeq p_{2!}((\mu \times \mathrm{id}) \circ \tilde{i})^* \Delta_! \bar{\mathbf{Q}}_\ell.$$

Now apply proper base change again with the cartesian squares:

$$\begin{array}{ccccc} \mathbf{A}^1 & \xrightarrow{i'} & \mathbf{A}^1 \times \mathbf{A}^1 & \xrightarrow{\mu} & \mathbf{A}^1 \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle \gamma_\mu} & & \downarrow{\scriptstyle \Delta} \\ \mathbf{A}^1 \times \mathbf{A}^1 & \xrightarrow{\tilde{i}} & \mathbf{A}^1 \times \mathbf{A}^1 \times \mathbf{A}^1 & \xrightarrow{\mu \times \mathrm{id}} & \mathbf{A}^1 \times \mathbf{A}^1 \end{array}$$

where $i'(x) = (x,0)$. This gives

$$i^*K \simeq p_{2!}((\mu \times \mathrm{id}) \circ \tilde{i})^* \Delta_! \bar{\mathbf{Q}}_\ell \simeq (p_2 \circ i')_! \bar{\mathbf{Q}}_\ell$$

But now $p_2 i'(x) = 0$. So $i^*K \simeq \delta_! \bar{\mathbf{Q}}_\ell[-2](-1)$.

3. Fourier transform.

3.1. Define $\mathrm{FT}\colon E(\mathbf{A}^1) \to E(\mathbf{A}^1)$ by:

$$\mathrm{FT}(K) = p_{2!}(p_1^*K \overset{+}{\otimes} \mu^*\mathbb{E})[1].$$

Sanity check:

$$\mathrm{FT}(\mathbb{1}) = p_{2!}(p_1^*\mathbb{1} \overset{+}{\otimes} \mu^*\mathbb{E})[1] \simeq p_{2!}(\mathbb{1} \overset{+}{\otimes} \mu^*\mathbb{E})[1] \simeq p_{2!}\mu^*\mathbb{E}[1] \simeq \delta_!\mathbb{1}[-1](-1)$$

by orthogonality above.

Theorem 3.2. *Let $a\colon \mathbf{A}^1 \to \mathbf{A}^1$ be the map $x \mapsto -x$. Then we have a canonical isomorphism:*

$$\mathrm{FT} \circ \mathrm{FT}(K) \simeq a^*K(-1).$$

Proof. We have cartesian squares:

$$\begin{array}{ccc} \mathbf{A}^1 \times \mathbf{A}^1 \times \mathbf{A}^1 & \xrightarrow{\alpha} & \mathbf{A}^1 \times \mathbf{A}^1 \\ {\scriptstyle p_{13}}\downarrow & & \downarrow{\scriptstyle p_2} \\ \mathbf{A}^1 \times \mathbf{A}^1 & \xrightarrow{\sigma} & \mathbf{A} \end{array} \qquad \begin{array}{ccccc} \mathbf{A}^1 & \xrightarrow{\Delta} & \mathbf{A}^1 \times \mathbf{A}^1 & \xrightarrow{a \times \mathrm{id}} & \mathbf{A}^1 \times \mathbf{A}^1 \\ \downarrow & & & & \downarrow{\scriptstyle \sigma} \\ \mathrm{Spec}(R) & & \xrightarrow[\delta]{} & & \mathbf{A}^1 \end{array}$$

$$\begin{array}{ccc} \mathbf{A}^1 \times \mathbf{A}^1 \times \mathbf{A}^1 & \xrightarrow{p_{12}} & \mathbf{A}^1 \times \mathbf{A}^1 \\ {\scriptstyle p_{23}}\downarrow & & \downarrow{\scriptstyle p_2} \\ \mathbf{A}^1 \times \mathbf{A}^1 & \xrightarrow{p_1} & \mathbf{A}^1 \end{array}$$

where Δ is the diagonal, $\delta\colon \mathrm{Spec}(R) \to \mathbf{A}^1$ is the zero section, $\sigma(x,y) = x+y$, and $\alpha(x,y,z) = (y, x+z)$. The maps p_{12}, etc. are projections to the indicated factors. Now we just compute:

$$
\begin{aligned}
\mathrm{FT}\circ\mathrm{FT}(K) &= p_{2!}(p_1^* p_{2!}(p_1^* K \overset{+}{\otimes} \mu^*\mathbb{E}) \overset{+}{\otimes} \mu^*\mathbb{E})[2] && \text{(by definition)}\\
&\simeq p_{2!}(p_{23!}p_{12}^*(p_1^* K \overset{+}{\otimes} \mu^*\mathbb{E}) \overset{+}{\otimes} \mu^*\mathbb{E})[2] && \text{(proper base change)}\\
&\simeq p_{2!}p_{23!}(p_{12}^*(p_1^* K \overset{+}{\otimes} \mu^*\mathbb{E}) \overset{+}{\otimes} p_{23}^*\mu^*\mathbb{E})[2] && \text{(projection formula)}\\
&\simeq p_{2!}p_{13!}(p_{12}^*p_1^* K \overset{+}{\otimes} p_{12}^*\mu^*\mathbb{E} \overset{+}{\otimes} p_{23}^*\mu^*\mathbb{E})[2] && (p_2\circ p_{23} = p_2 \circ p_{13})\\
&\simeq p_{2!}p_{13!}(p_{13}^*p_1^* K \overset{+}{\otimes} p_{12}^*\mu^*\mathbb{E} \overset{+}{\otimes} p_{23}^*\mu^*\mathbb{E})[2] && (p_1\circ p_{12} = p_1 \circ p_{13})\\
&\simeq p_{2!}p_{13!}(p_{13}^*p_1^* K \overset{+}{\otimes} \alpha^*\mu^*\mathbb{E})[2] && \text{(additivity)}\\
&\simeq p_{2!}(p_1^* K \overset{+}{\otimes} p_{13!}\alpha^*\mu^*\mathbb{E})[2] && \text{(projection formula)}\\
&\simeq p_{2!}(p_1^* K \overset{+}{\otimes} \sigma^* p_{2!}\mu^*\mathbb{E})[2] && \text{(proper base change)}\\
&\simeq p_{2!}(p_1^* K \overset{+}{\otimes} \sigma^* \delta_! \mathbb{1}_{\mathrm{Spec}(R)})(-1) && \text{(orthogonality)}\\
&\simeq p_{2!}(p_1^* K \overset{+}{\otimes} (a\times \mathrm{id})_! \Delta_! \mathbb{1}_{\mathbf{A}^1})(-1) && \text{(proper base change)}\\
&\simeq p_{2!}(p_1^* K \overset{+}{\otimes} (a\times \mathrm{id})^* \Delta_! \mathbb{1}_{\mathbf{A}^1})(-1) && (a\circ a = \mathrm{id})\\
&\simeq p_{2!}(a\times \mathrm{id})^*(p_1^* a^* K \overset{+}{\otimes} \Delta_! \mathbb{1}_{\mathbf{A}^1})(-1) && (a\circ a = \mathrm{id})\\
&\simeq p_{2!}(p_1^* a^* K \overset{+}{\otimes} \Delta_! \mathbb{1}_{\mathbf{A}^1})(-1) && (a\circ a = \mathrm{id} \text{ and } p_2\circ(a\times\mathrm{id}) = p_2)\\
&\simeq p_{2!}\Delta_!(\Delta^* p_1^* a^* K \overset{+}{\otimes} \mathbb{1}_{\mathbf{A}^1})(-1) && \text{(projection formula)}\\
&\simeq p_{2!}\Delta_! a^* K(-1) && (p_1\circ\Delta = \mathrm{id})\\
&\simeq a^*K(-1) && (p_2\circ\Delta = \mathrm{id}).
\end{aligned}
$$

□

4. **To-Do.** Show $E(X)$ is equivalent to (but not embedded as) the *subcategory* of $D(X\times\mathbf{A}^1)$ given by objects satisfying $p_!K = 0$. Should be a 'formal' argument via the projector $j_*\bar{\mathbf{Q}}_\ell$.

4.1. *Side remark.* In hindsight, it might have been "better" to work with $\mathbf{G}_a$ (or unipotent groups) instead of $\mathbf{A}^1$.