LINE FIELDS ON MANIFOLDS

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A line field on a smooth manifold M is a section of the projectivized tangent bundle. If such a section is only given over $M - \{x_1, \dots, x_k\}$, for some finite number of points in M, then it is called a line field with singularities at the x_i. Every vector field induces a line field. However, not every line field arises this way.

Theorem 1. [Hopf, Theorem 2.2]*Let s be a line field with singularities $x_1, \dots, x_k$ on a compact orientable surface Σ. Then*

$$\sum_i (\text{local degree of } s \text{ at } x_i) = 2\chi(\Sigma),$$

where χ denotes the Euler characteristic.

'Local degree' is defined below. Note that the definition is twice that of [Hopf]. A higher dimensional generalization of Hopf's result is proved in [CG]. The precise attribution for this more general statement is a convoluted matter since the authors of [CG] correct some misstatements in the literature. M. Grant has informed me that the first complete proof of said generalization appears to be due to K. Jänich [J, §1, §2]. However, I have not made any serious attempt to go down the attribution rabbit hole. See [CG, §1] for a more complete history. Anyway, the purpose of the present note is to give a simple proof of the main result of [CG] (see Corollary 4). Along the way, a mild generalization (Theorem 3) is provided at no extra charge.

Let $E \to M$ be an oriented rank n vector bundle over an oriented manifold of even dimension n. Write $PE \to M$ for its projectivization. Notice that the fibre of PE is the *odd dimensional* real projective space $\mathbf{RP}^{n-1}$. Now suppose s is a section of $PE \to M$ over a punctured neighborhood of a point $x \in M$. Let D be a small coordinate disk centered at x over which PE is trivial. Then D inherits an orientation from M. Orient $\mathbf{RP}^{n-1}$ so that the isomorphism $PE|_D \cong D \times \mathbf{RP}^{n-1}$ is orientation preserving. The *local degree* of the section s at x is defined to be the degree of the composite map:

$$\partial \bar{D} \xrightarrow{s} PE|_{\bar{D}} \xrightarrow{\cong} \bar{D} \times \mathbf{RP}^{n-1} \xrightarrow{\text{projection}} \mathbf{RP}^{n-1},$$

where $\bar{D}$ is the closure of D, and $\partial \bar{D}$ is the boundary (diffeomorphic to S^{n-1}).

Proposition 2. *Let $E \to M$ be an oriented vector bundle of rank n on a compact oriented manifold M of even dimension n. Let $\pi \colon PE \to M$ be its projectivization. If s and t are sections of π over $M - \{x_1, \dots, x_k\}$ and $M - \{y_1, \dots, y_l\}$, respectively, then*

$$\sum_i (\text{local degree of } s \text{ at } x_i) = \sum_j (\text{local degree of } t \text{ at } y_j).$$

Proof. The de Rham cohomology of the fibre (an odd dimensional real projective space) vanishes except for in degree 0 and $n-1$. Moreover, orientability of the vector bundle gives a consistent choice of fundamental class in the cohomology of each fibre of π. Consequently, such a fundamental class is transgressive (see [BT, §11, §18]).

In other words, there exists a global differential form ψ on PE that restricts to the fundamental class in the cohomology of each fibre, and is such that $d\psi = -\pi^*\tau$ for some form τ on M (here d is the exterior derivative). Write D_i for a coordinate disk centred at x_i. Then

$$\begin{aligned}\int_{M-\cup_i D_i} \tau &= \int_{M-\cup_i D_i} s^*\pi^*\tau \\ &= -\int_{M-\cup_i D_i} s^* d\psi \\ &= \sum_i \int_{\partial \bar{D}_i} s^*\psi \qquad \text{(by Stokes' Theorem).}\end{aligned}$$

Taking the limit as the radius of the D_i tends to 0 yields

$$\int_M \tau = \sum_i (\text{local degree of } s \text{ at } x_i).$$

Similarly for the section t. As τ is independent of the sections, the result follows. □

Theorem 3. *Let $E \to M$ be an oriented rank n vector bundle on a compact orientable manifold M of even dimension n. Let $E \to M$ be its projectivization. If s is a section of PE over $M - \{x_1, \ldots, x_k\}$, then*

$$\sum_i (\text{local degree of } s \text{ at } x_i) = 2\int_M e(E),$$

where $e(E)$ is the Euler class of E.

Proof. In view of Proposition 2 we may replace s by a section that comes from a section t of $E \to M$ with finitely many zeroes $\{y_1, \ldots, y_l\}$. Such a t always exists ([BT, Proposition 11.14]). It is known that (for instance, see [BT, Theorem 11.17]):

$$\sum_j (\text{local degree of } t \text{ at } y_j) = \int_M e(E).$$

Note that the 'local degree' above is that of the section of the vector bundle (as opposed to its projectivization), defined in the usual way (see [BT, §11]). This implies the desired result. □

Corollary 4. *Let s be a line field with singularities $\{x_1, \ldots, x_k\}$ on a compact oriented manifold of even dimension. Then*

$$\sum_i (\text{local degree of } s \text{ at } x_i) = 2\chi(M).$$

References

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