Monodromy triangle for nearby cycles

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We start with the standard setup

$$\begin{tikzcd} X_0 \arrow[r, hook, "i"] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \arrow[dr, phantom, "\square"] & X^* \arrow[l, hook', "j"'] \arrow[d] \\ \{0\} \arrow[r, hook] & \mathbb{A}^1 & \mathbb{A}^1 - \{0\} \arrow[l, hook'] \end{tikzcd}$$

$$\left.\begin{array}{l} \Psi : D(X) \longrightarrow D(X_0) \\ \Phi : D(X) \longrightarrow D(X_0) \end{array}\right\}$$ unipotent parts of nearby and vanishing cycles, resp.

shift convention: both Ψ and Φ are perverse t-exact.

We have natural transformations:

$$\mathrm{can} : \Psi \longrightarrow \Phi$$

$$\mathrm{var} : \Phi \longrightarrow \Psi(-1)$$ Tate twist

$$N = \mathrm{var} \circ \mathrm{can} \qquad (\log \text{ of monodromy})$$

These fit into distinguished triangles:

$$\Psi \xrightarrow{\mathrm{can}} \Phi \longrightarrow i^* \longrightarrow \Psi[1]$$

$$i^! \longrightarrow \Phi \xrightarrow{\mathrm{var}} \Psi(-1) \longrightarrow i^![1]$$

with the following compatibility: the composite map

$$i^! \longrightarrow \Phi \longrightarrow i^*$$

coincides with the composite map

$$i^! \xrightarrow{\sim} i^* i_* i^! \xrightarrow{\text{adjunction}} i^* \qquad (A)$$

By the octahedron axiom (so called TR(4)), there exists a distinguished triangle $i^* \longrightarrow \mathrm{cone}(N) \longrightarrow i^![1] \xrightarrow{\alpha} i^*[1]$ such that α coincides with $i^![1] \longrightarrow \Phi[1] \longrightarrow i^*[1]$.

Thus, ~~this~~ $\alpha[-1]$ coincides with (A). Consequently:

$$\mathrm{cone}(N) \simeq \mathrm{cone}(i^! \longrightarrow i^*)$$

Now apply i^* to the canonical distinguished triangle:

$$i_* i^! \longrightarrow \mathrm{id} \longrightarrow j_* j^* \longrightarrow i_* i^![1]$$

to get the distinguished triangle:

$$i^! \longrightarrow i^* \longrightarrow i^* j_* j^* \longrightarrow i^![1]$$

i.e., $\mathrm{cone}(N) \simeq i^* j_* j^*$ and we have the monodromy triangle:

$$\Psi \xrightarrow{N} \Psi(-1) \longrightarrow i^* j_* j^* \longrightarrow \Psi[1]$$

$\square$