HODGE-GROTHENDIECK CLASSES AND MONODROMY INVARIANTS OF NEARBY CYCLES SHEAVES

R. VIRK

This note was prompted by a question of G. Williamson [W] (paraphrasing): *"How much information about nearby cycles can we deduce without knowing the defining function?"*. The idea being that the associated graded of the monodromy filtration[1] on (unipotent) nearby cycles is determined by its primitive part. The latter should be completely determined by the central fibre.

We pursue this in the Hodge theoretic context. Theorem **4.2** shows that the Hodge-Grothendieck class of (the unipotent part of) nearby cycles sheaves is independent of the defining equation. Theorem **5.2** is a generalized local invariant cycles result (a variant of [D, Théorème 3.6.1]). No claims to orginality are being made.

1. Notation.

1.1. We write $\mathscr{M}(X)$ for the category of mixed Hodge modules on a variety[2] X, and $\mathscr{D}(X)$ for its bounded derived category. The cohomology functors associated to the evident t-structure (with heart $\mathscr{M}(X)$) are denoted ${}^pH^i\colon \mathscr{D}(X) \to \mathscr{M}(X)$. The d-th Tate twist will be denoted by (d). Functors on derived categories will always be derived. I.e., we write f_* instead of Rf_*, etc.

1.2. Let $f: X \to \mathbf{A}^1$ be a morphism of varieties. Set $X_0 = f^{-1}(0)$ and $X^* = X - X_0$. We write $i: X_0 \to X$ and $j: X^* \to X$ for the inclusions.

$$\begin{array}{ccccc} X_0 & \xrightarrow{i} & X & \xleftarrow{j} & X^* \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ \{0\} & \longrightarrow & \mathbf{A}^1 & \longleftarrow & \mathbf{A}^1 - \{0\} \end{array}$$

The unipotent part of the nearby cycles functor associated to f is denoted:

$$\psi_f^u\colon \mathscr{D}(X) \to \mathscr{D}(X_0)$$

Shift convention is that $\psi_f^u[-1]$ is t-exact. We write:

$$N\colon \psi_f^u \to \psi_f^u(-1)$$

for the log of the unipotent part of monodromy.

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[1] In the sense of [D, 1.6].

[2] 'Variety' = 'separated scheme of finite type over Spec($\mathbf{C}$)'.

2. Preliminaries.

2.1. For $M \in \mathscr{D}(X^*)$ it is convenient to set:

$$ {}^p\Psi^{\mathrm{u}}_f(M) = \psi^u_f(j_* M)[-1]. $$

In particular, ${}^p\Psi^{\mathrm{u}}_f$ restricts to a t-exact functor $\mathscr{M}(X^*) \to \mathscr{M}(X_0)$.[3]

2.2. We have a canonical distinguished triangle (see [S88, Remark 5.2.2]):

$$ {}^p\Psi^{\mathrm{u}}_f(M) \xrightarrow{N} {}^p\Psi^{\mathrm{u}}_f(M)(-1) \to i^* j_* M \xrightarrow{[1]} \tag{2.2.1} $$

For $M \in \mathscr{M}(X^*)$ this yields an exact sequence:

$$ 0 \to {}^pH^{-1}(i^* j_* M) \to {}^p\Psi^{\mathrm{u}}_f(M) \xrightarrow{N} {}^p\Psi^{\mathrm{u}}_f(M)(-1) \to {}^pH^0(i^* j_* M) \to 0 \tag{2.2.2} $$

Proposition 2.3. *Let $M \in \mathscr{M}(X^*)$. Let $j_{!*}(M) \in \mathscr{M}(X)$ be the intermediate extension of M to X. Then we have canonical isomorphisms:*

(i) $\ker(N) \simeq i^* j_{!*}(M)[-1]$
(ii) $\mathrm{coker}(N) \simeq i^! j_{!*}(M)[1]$

Proof. We only show (i). The proof of (ii) is similar. Apply i^* to the canonical distinguished triangle:

$$ i_* i^!(j_{!*}(M)) \to j_{!*}(M) \to j_* j^*(j_{!*}(M)) \xrightarrow{[1]} $$

to get the distinguished triangle:

$$ i^! j_{!*}(M) \to i^* j_{!*}(M) \to i^* j_* M \xrightarrow{[1]} $$

This yields the exact sequence:

$$ {}^pH^{-1}(i^! j_{!*}(M)) \to {}^pH^{-1}(i^* j_{!*}(M)) \to {}^pH^{-1}(i^* j_* M) \to {}^pH^0(i^! j_{!*}(M)). $$

As $i^!$ is left t-exact, the left most term must vanish. Additionally, since $j_{!*}(M)$ is the intermediate extension of M, the right most term must also vanish. In other words, $\ker(N) \simeq {}^pH^{-1}(i^* j_{!*}(M))$. On the other hand, ${}^pH^k(i^* j_{!*}(M)) = 0$ for $k \neq -1$. □

3. Weights.

3.1. Mixed Hodge modules come equipped with a finite increasing filtration (the weight filtration) which we denote by $W_\bullet$. The associated graded is denoted $\mathrm{Gr}^W_\bullet$. Morphisms in $\mathscr{M}(X)$ are *strictly* compatible with $W_\bullet$ [S89, 1.5].

3.2. An object $M \in \mathscr{D}(X)$ is said to have weights $\leq n$ (resp. $\geq n$) if $\mathrm{Gr}^W_k {}^pH^i(M) = 0$ for $k > i+n$ (resp. $k < i+n$). The object M is called pure of weight n if $\mathrm{Gr}^W_k {}^pH^i(M) = 0$ for $k \neq i + n$.

[3] Our ${}^p\Psi^{\mathrm{u}}_f$ is denoted $\psi_{f,1}$ in [S89].

3.3. Let $M \in \mathscr{M}(X^*)$. Then:

$$N W_k {}^p\Psi_f^{\mathrm{u}}(M) \subset W_{k-2} {}^p\Psi_f^{\mathrm{u}}(M)(-1),$$

for each k. Additionally, if M is pure of weight n, then:

$$N^k : \mathrm{Gr}^W_{n-1+k} {}^p\Psi_f^{\mathrm{u}}(M) \xrightarrow{\sim} \mathrm{Gr}^W_{n-1-k} {}^p\Psi_f^{\mathrm{u}}(M)(-k)$$

is an isomorphism for each $k \geq 0$ (see [S89, 1.19] and [D, 1.6]). Further, we have an isomorphism:

$$\mathrm{Gr}^W_k \ker(N) \simeq \ker(N : \mathrm{Gr}^W_k {}^p\Psi_f^{\mathrm{u}}(M) \to \mathrm{Gr}^W_{k-2} {}^p\Psi_f^{\mathrm{u}}(M)(-1)).$$

Consequently, for M pure of weight n, we have an isomorphism:

$$\mathrm{Gr}^W_k {}^p\Psi_f^{\mathrm{u}}(M) \simeq \bigoplus_{\substack{m \geq |n-1-k|, \\ m = n-1-k \pmod 2}} \mathrm{Gr}^W_{n-1-m} \ker(N)((n-1-m-k)/2) \tag{3.3.1}$$

4. Hodge-Grothendieck classes.

4.1. Let $K_0(X_0)$ denote the Grothendieck group of $\mathscr{D}(X_0)$ (equivalently $\mathscr{M}(X_0)$).

Theorem 4.2. *Let $f, g : X \to \mathbf{A}^1$ be morphisms of varieties. If $f^{-1}(0)_{\mathrm{red}} = g^{-1}(0)_{\mathrm{red}}$, then ψ_f^u and ψ_g^u define the same map on Grothendieck groups. I.e., in $K_0(X_0)$:*

$$[\psi_f^u(M)] = [\psi_g^u(M)],$$

for each $M \in \mathscr{D}(X)$

Proof. It suffices to show $[{}^p\Psi_f^{\mathrm{u}}(M)] = [{}^p\Psi_g^{\mathrm{u}}(M)]$ for $M \in \mathscr{M}(X^*)$. We may also assume that M is pure of weight n. By Proposition 2.3(i) and (3.3.1):

$$\begin{aligned}[{}^p\Psi_f^{\mathrm{u}}(M)] &= \sum_k \sum_{\substack{m \geq |n-1-k|, \\ m = n-1-k \pmod 2}} [\mathrm{Gr}^W_{n-1-m} i^* j_{!*} M[-1]((n-1-m-k)/2)] \\ &= [{}^p\Psi_g^{\mathrm{u}}(M)]\end{aligned}$$

□

Remark 4.3. If only the ordinary Grothendieck class of ψ_f^u is of interest (i.e., working with constructible sheaves as opposed to mixed Hodge modules), then Hodge theory may be completely avoided as follows. Replace $W_\bullet$ by the monodromy filtration associated to N and argue exactly as above. The identity (3.3.1) holds. The only extra ingredient needed in the proof of Theorem 4.2 is that the induced filtration on $i^* j_{!*}(M)$ is independent of f.[4] A purely topological demonstration of this is the main result of [ELM].[5]

[4]This is the heart of the matter in the proof of Theorem 4.2. It is somewhat obscured by the expository choice of not making it totally explicit that the monodromy filtration and weight filtration coincide.
[5]The point is that the monodromy filtration induced from nearby cycles coincides with that induced from Verdier specialization. Verdier specialization does not depend on f.

5. Invariant cycles.

5.1. The adjunction map $M \to j_* j^* M$ along with (2.2.1) yields canonical maps:

$$i^* M \to i^* j_* j^* M \to \psi_f^u(M).$$

Theorem 5.2 (Local invariant cycles). *Let $M \in \mathscr{D}(X)$ be pure. If $f : X \to \mathbf{A}^1$ is proper, then the sequence of maps:*

$$H^k(X_0; i^* M) \to H^k(X_0; \psi_f^u(M)) \xrightarrow{N} H^k(X_0; \psi_f^u(M))(-1)$$

is exact for each k.

Proof. Say M is pure of weight n. As pushforward along a proper map preserves weights [S89, 1.8], the weight filtration on $H^k(X_0; \psi_f^u(M))$ is the monodromy filtration[6] centered at $n+k$ [S88, (5.3.4.2)], [S89, 1.19]. In particular, ker(N) has weights $\leq n+k$. From (2.2.1) we infer that ker(N) is the image of:

$$H^k(X_0; i^* j_* j^* M) \to H^k(X_0; \psi_f^u(M)).$$

Thus, it suffices to show $H^k(X_0; i^* M) \to H^k(X_0; i^* j_* j^* M)$ is surjective on weights $\leq n+k$. The map $i^* M \to i^* j_* j^* M$ fits into a distinguished triangle:

$$i^! M \to i^* M \to i^* j_* j^* M \xrightarrow{[1]}$$

This yields an exact sequence:

$$H^k(X_0; i^* M) \to H^k(X_0; i^* j_* j^* M) \to H^{k+1}(X_0; i^! M).$$

As $i^!$ does not lower weights [S89, 1.7], the right most term has weights $\geq n+k+1$. Hence, $H^k(X_0; i^* M) \to H^k(X_0; i^* j_* j^* M)$ must be surjective on weights $\leq n+k$. □

Remark 5.3. If X is rationally smooth, then taking M to be the constant sheaf on X (with trivial Hodge structure)[7] recovers the classical local invariant cycles theorem.

Remark 5.4. Using an argument similar to the proof above, the exact sequence of Theorem 5.2 may be extended to a generalized Clemens-Schmid exact sequence.

References

[D] P. Deligne, *La conjecture de Weil: II*, Publ. I.H.É.S **52** (1980).

[ELM] F. El Zein, D.T. Lê, L. Migliorini, *A topological construction of the weight filtration*, Manuscripta Math. **133** (2010).

[S88] M. Saito, *Modules de Hodge polarisables*, Publ. RIMS **24** (1988).

[S89] M. Saito, *Introduction to mixed Hodge modules*, Astérisque **179-180** (1989).

[W] G. Williamson, *Nearby cycles without a function*, https://mathoverflow.net/q/389689.

[6] In the sense of [D, 1.6]

[7] More precisely, $\mathscr{M}(\mathrm{Spec}(\mathbf{C}))$ is the category of polarizable $\mathbf{Q}$-mixed Hodge structures [S89, 1.4]. Let $\mathbf{Q}^H$ be the trivial 1-dimensional, weight 0 Hodge structure. Let $a : X \to \mathrm{Spec}(\mathbf{C})$ be the structure map. Take $M = a^* \mathbf{Q}^H$.