MOTIVIC SPLITTING PRINCIPLE

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1. **Introduction.** Fix a field k and a commutative ring Λ (with 1). Assume either that k is perfect and admits resolution of singularities, or that k is arbitrary and the exponential characteristic[1] of k is invertible in Λ. Write $DM(k;\Lambda)$ for the triangulated category of motives with Λ-coefficients (see §2).

We write 'scheme' in lieu of 'separated scheme of finite type over k'. For a group scheme G, a *G-torsor* will mean a faithfully flat G-invariant morphism $X \to Y$, such that the canonical morphism $G \times X \to X \times_Y X$ is an isomorphism. In this situation, we say that the *quotient* of the G-action on X exists, and we set $X/G = Y$. The *trivial torsor* over a scheme X is the projection $G \times X \to X$, with G acting on $G \times X$ via multiplication on G. A *reductive group* will mean a smooth affine group scheme G such that every smooth connected unipotent subgroup of $G \times_k \bar{k}$ is trivial, where $\bar{k}$ is the algebraic closure of k.

Theorem 1.1. *Let G be a connected split reductive group, and let $B \subset G$ be a Borel subgroup. Let X be a scheme with G-action. Let $t(G)$ denote the torsion index of G. If $t(G)$ is invertible in Λ, then there is an isomorphism*

$$M^c(X_{hB}) \simeq M^c(X_{hG}) \otimes M^c(G/B)$$

in $DM(k;\Lambda)$ that commutes with smooth pullbacks and localization triangles.

Here $M^c(X_{hG})$, $M^c(X_{hB})$ denote the G-equivariant and B-equivariant (Borel-Moore) motives, respectively, of X (in the sense of [T4]; see §3).

The proof of Theorem 1.1 is essentially the same as that of the analogous statement for the cohomology of topological spaces (for instance, compare with [M] and [T3, Theorem 16.1]). No claim to originality is being made.

Example 1.2. If $X = \mathrm{Spec}(k)$, then Theorem 1.1 states

$$M^c(BB) \simeq M^c(BG) \otimes M^c(G/B),$$

where $M^c(BB)$ and $M^c(BG)$ are the motives of the classifying spaces of B and G, respectively (see [T4] and §3). This is a motivic analogue of the usual splitting principle (working with groups over the complex numbers say):

$$H^*(BB) \simeq H^*(BG) \otimes H^*(G/B),$$

where cohomology is with coefficients in a ring Λ in which p is invertible for all primes p such that $H^*(G;\mathbf{Z})$ has p-torsion.

[1] If $\mathrm{char}(k) = 0$, then the exponential characteristic of k is 1. If $\mathrm{char}(k) > 0$, then the exponential characteristic of k is $\mathrm{char}(k)$.

2. **Motives.** The category $DM(k;\Lambda)$ is the monoidal triangulated category of Nisnevich motivic spectra over k, obtained by applying $\mathbf{A}^1$-localization and $\mathbf{P}^1$-stabilization to the derived category of (unbounded) complexes of Nisnevich sheaves with transfer [CD1, Definition 11.1.1].[2] The primary references for the properties of $DM(k;\Lambda)$ that we use are [V], [CD1] and [CD2]. The assumption that k is perfect and admits resolution of singularities stems from the treatment in [V]. The alternate assumption that k is arbitrary, but the exponential characteristic is invertible in Λ, stems from [CD2, Proposition 8.1] and [K]. These articles, amongst other things, extend the constructions of [V] under this alternate hypothesis.

There is a covariant functor $X \mapsto M^c(X)$ from the category of schemes and proper morphisms to $DM(k;\Lambda)$ (see [V, §2.2]; alternatively, in the notation of [CD1], we have $M^c(X) = a_* a^! \Lambda$, where $a\colon X \to \mathrm{Spec}(k)$ is the structure morphism). The functor $M^c(X)$ behaves like a Borel-Moore homology theory in the following sense. If $f\colon X \to Y$ is a smooth morphism with fibres of dimension r (the morphism f is allowed to have some fibres empty), then there is a map (compatible with composition of morphisms):

$$f^*\colon M^c(Y)(r)[2r] \to M^c(X),$$

where (j) denotes tensoring with the j-th Tate twist (see [CD1, (11.1.2.2]) and $[i]$ denotes the i-th shift functor (available in any triangulated category). If f is a vector bundle, then f^* is an isomorphism.

Let $i\colon Z \hookrightarrow X$ be a closed immersion, and let $j\colon U \hookrightarrow X$ be the open immersion of the complement $U = X - Z$. Then i_* and j^* fit into a canonical distinguished triangle [V, §2.2], [CD2, Corollary 5.9, Theorem 5.11], the *localization triangle*,

$$M^c(Z) \xrightarrow{i_*} M^c(X) \xrightarrow{j^*} M^c(U) \xrightarrow{\partial_i}$$

The category $DM(k;\Lambda)$ is a symmetric monoidal triangulated category, and

$$M^c(X \times Y) = M^c(X) \otimes M^c(Y).$$

The motive $M^c(\mathrm{Spec}(k))$ is the unit object. Although notationally abusive, it is convenient to set

$$\Lambda = M^c(\mathrm{Spec}(k)).$$

Let $H^i_M(X;\Lambda(j))$ denote the motivic cohomology groups of X, as defined in [CD1, §11.2]. These are contravariant functors from the category of schemes to Λ-modules. By [CD1, Example 11.2.3], if X is smooth and equidimensional, then motivic cohomology determines the Chow ring $CH^*(X)$ of X:

$$H^{2j}_M(X;\Lambda(j)) = CH^j(X) \otimes_{\mathbf{Z}} \Lambda.$$

For an arbitrary scheme X, each $e \in H^i(X;\Lambda(j))$ determines a canonical map

$$e\cap\colon M^c(X)(-j)[-i] \to M^c(X).$$

[2] The category $DM^{eff}_{-}(k;\Lambda)$ of [V] embeds into $DM(k;\Lambda)$ as a full and faithful subcategory [CD1, Example 11.1.3]. Utilizing $DM(k;\Lambda)$ (instead of $DM^{eff}_{-}(k;\Lambda)$) is dictated by the need for arbitrary direct sums and compact generation [V, Corollary 3.5.5]. Both of these properties are required to make sense of equivariant motives (see [T4] and §3).

Example 2.1. Let $f\colon L \to X$ be a line bundle. Write $i\colon X \hookrightarrow L$ for the zero section. Let $c_1(L) \in H^2_M(X;\Lambda(1))$ be the first Chern class of L [CD1, Definition 11.3.2]. Then $c_1(L)\cap$ is the composition

$$M^c(X)(-1)[-2] \xrightarrow{i_*} M^c(L)(-1)[-2] \xrightarrow{f^{*-1}} M^c(X).$$

3. **Equivariant motives, following B. Totaro.** Let G be an affine group scheme. Let X be a scheme with G-action. Let $\cdots \twoheadrightarrow V_2 \twoheadrightarrow V_1$ be a sequence of surjections of G-equivariant vector bundles over X. Write n_i for the rank of V_i. Let $U_i \subset V_i$ be a G-stable open subscheme such that $V_{i+1} - U_{i+1}$ is contained in the inverse image of $V_i - U_i$, and such that the quotient U_i/G exists. Assume that the codimension of $V_i - U_i$ goes to infinity with i. The *equivariant motive* $M^c(X_{hG})$ is defined to be the homotopy limit of the sequence

$$\cdots \to M^c(U_2/G)(-n_2)[-2n_2] \to M^c(U_1/G)(-n_1)[-2n_1].$$

The definition of $M^c(X_{hG})$ is independent (up to a not necessarily unique isomorphism) of all the choices involved [T4, Theorem 8.5]. If X satisfies any of the conditions of [EG, Proposition 23], then such vector bundles exist (also see [T1, Remark 1.4] and [T4, §8]). Whenever we speak of $M^c(X_{hG})$, we implicitly, and without further comment, assume that such vector bundles exist (including in the statement of Theorem 1.1).

Since $M^c(X_{hG})$ is defined as a (homotopy) limit of ordinary motives $M^c(U_i/G)$, the functorial properties of ordinary motives (pullback, pushforward, localization triangles, etc.) extend to the equivariant setup. By construction, $M^c(X_{hG})$ satisfies equivariant descent: if the quotient of the G-action on X exists, then

$$M^c(X_{hG}) \simeq M^c(X/G).$$

We set

$$M^c(BG) = M^c(\mathrm{Spec}(k)_{hG}).$$

Example 3.1. Let V_i be the direct sum of i-copies of the natural 1-dimensional representation of $\mathbf{G}_m$. Let $U_i = V_i - \{0\}$. Then $U_i/\mathbf{G}_m \simeq \mathbf{P}^{i-1}$. We have [T4, Lemma 8.7]:

$$M^c(B\mathbf{G}_m) \simeq \prod_{i\leq -1} \Lambda(i)[2i].$$

4. **Restriction to a subgroup.** Let G be an affine group scheme. Let $H \subset G$ be a closed subgroup. If the quotient X/G exists, then the quotient X/H exists. If G is smooth, then we have a pullback

$$M^c(X/G)(\dim(G/H))[2\dim(G/H)] \to M^c(X/H).$$

This family of pullbacks, one for each such X, yields a map, *restriction*,

$$\mathrm{res}^H_G\colon M^c(Y_{hG})(\dim(G/H))[2\dim(G/H)] \to M^c(Y_{hH}),$$

for any scheme Y with G-action. Restriction commutes with smooth pullbacks and localization triangles.

5. **Chow ring of a classifying space.** Let G be an affine group scheme. Following [T1], define the Chow ring CH^*_G of the classifying space of G as follows. Let V be a representation of G over k. Let $U \subset V$ be an open subscheme such that the quotient U/G exists, and such that $V - U$ has codimension greater than i. Then $CH^i_G = CH^i(U/G)$. This definition is independent of all the choices involved [T1, Theorem 1.1] and gives a well-defined ring CH^*_G. It follows from the definition that each $e \in CH^i_G$ determines a canonical map

$$e\cap\colon M^c(X_{hG})(-i)[-2i] \to M^c(X_{hG}),$$

for a scheme X with an action of G.

By faithfully flat descent, each representation of G over k determines a vector bundle over the schemes U/G used to define CH^*_G. Consequently, each such representation has Chern classes in CH^*_G.

Example 5.1. Let T be a split torus. Let χ be a character of T, with first Chern class $c_1(\chi) \in CH^1_T$. Let X be a scheme with T-action. Then χ determines an equivariant line bundle $L_\chi \to X$. If the quotient X/T exists, then L_χ descends to a line bundle $\tilde{L}_\chi \to X/T$. In this situation, the map

$$c_1(\chi)\cap\colon M^c(X_{hT})(-1)[-2] \to M^c(X_{hT})$$

is the composition

$$M^c(X_{hT})(-1)[-2] \xrightarrow{\sim} M^c(X/T)(-1)[-2] \xrightarrow{c_1(\tilde{L}_\chi)\cap} M^c(X/T) \xrightarrow{\sim} M^c(X_{hT}),$$

where the first and last isomorphisms are taken to be inverse to each other.

6. **The torsion index.** Let G be a connected split reductive group over k. Let $B \subset G$ be a Borel subgroup. The *torsion index* of G is the smallest integer $t(G) \in \mathbf{Z}_{>0}$ such that the image of the map $CH^*_B \to CH^*(G/B)$ contains $t(G) \cdot CH^{\dim(G/B)}(G/B)$. The natural map,

$$CH^*_B \otimes_{\mathbf{Z}} \mathbf{Z}[t(G)^{-1}] \to CH^*(G/B) \otimes_{\mathbf{Z}} \mathbf{Z}[t(G)^{-1}],$$

is surjective. According to [G, Théorème 2], for any G-torsor $X \to Y$, there is a non-empty open subscheme $U \subset Y$ along with a finite étale morphism $V \to U$ of degree invertible in $\mathbf{Z}[t(G)^{-1}]$, such that X is trivial over V.[3]

Example 6.1. The group GL_n has torsion index 1.

7. **Proof of Theorem 1.1.** (Compare with [M] and the proof of [T3, Theorem 16.1]). Pick elements $e_1, \dots, e_n \in CH^*(BB) \otimes_{\mathbf{Z}} \Lambda$, of homogeneous degree, that restrict to a basis of $CH^*(G/B) \otimes_{\mathbf{Z}} \Lambda$. Write d_i for the degree of e_i. Set $d = \dim(G/B)$. For each e_i, consider the composition

$$M^c(X_{hG})(d-d_i)[2(d-d_i)] \xrightarrow{\mathrm{res}^B_G} M^c(X_{hB})(-d_i)[-2d_i] \xrightarrow{e_i\cap} M^c(X_{hB}).$$

Summing these, we obtain a map

$$\bigoplus_i M^c(X_{hG})(d-d_i)[2(d-d_i)] \to M^c(X_{hB}).$$

[3] The point is that, if one stays away from primes that divide $t(G)$, then all the challenges of 'étale descent' for equivariant Chow groups disappear. For further information on the torsion index, [T2] is highly recommended.

By the Bruhat decomposition, this may be rewritten as a map

$$\theta\colon M^c(X_{hG}) \otimes M^c(G/B) \to M^c(X_{hB}).$$

The map θ commutes with smooth pullbacks and localization triangles. We will show θ is an isomorphism. It suffices to demonstrate this under the assumption that the quotient X/G exists. Via the isomorphisms $M^c(X_{hG}) \simeq M^c(X/G)$ and $M^c(X_{hB}) \simeq M^c(X/B)$, the map θ yields a map

$$\theta_X\colon M^c(X/G) \otimes M^c(G/B) \to M^c(X/B).$$

If $X \to X/G$ is the trivial G-torsor, then θ_X is manifestly an isomorphism. In general, there exists a non-empty open subscheme $U \subset X/G$, along with a finite étale morphism $f\colon V \to U$ of degree invertible in $\mathbf{Z}[t(G)^{-1}]$, such that X pulled back to V is the trivial G-torsor (see §6). The map $f_* f^*\colon M^c(U) \to M^c(U)$ is the degree of f times the identity (as follows from [CD1, A.5 (6)] and [CD1, Proposition 11.2.5]). Consequently, θ_U is an isomorphism. Now let $Z = X - U$ be the closed complement (with reduced scheme structure say). Then, by virtue of the localization triangle, it suffices to show θ_Z is an isomorphism. This follows from an induction on dimension (the base case has been dealt with by the above considerations).

8. **A complement.** The Chow ring CH_B^* acts on $M^c(X_{hB})$. Under the isomorphism

$$M^c(X_{hB}) \simeq M^c(X_{hG}) \otimes M^c(G/B),$$

this action on the right hand side is the action of CH_B^* on $M^c(G/B)$. Further, if we let $d = \dim(G/B)$ (this is being done for purely typographical reasons), then the restriction map,

$$\mathrm{res}_G^B\colon M^c(X_{hG})(d)[2d] \to M^c(X_{hB}),$$

also has a simple description under this isomorphism. It is the composition

$$M^c(X_{hG})(d)[2d] \xrightarrow{\sim} M^c(X_{hG}) \otimes \Lambda(d)[2d] \xrightarrow{\mathrm{id}\otimes a^*} M^c(X_{hG}) \otimes M^c(G/B),$$

where $a\colon G/B \to \mathrm{Spec}(k)$ is the structure morphism.

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