Contents

1. Conventions

1.1. 'Map' and 'morphism' will be used interchangeably.

1.2. 'Functorial' and 'canonical' are synonyms for 'natural transformation of functors'.

1.3. Topological spaces are assumed to be locally compact.

1.4. 'Variety' = 'separated scheme of finite type over Spec($\mathbf{C}$)'.

1.5. 'Smooth map of topological spaces' means 'topological submersion' (see [KS, §3.3]). A morphism of varieties that is smooth in the sense of algebraic geometry is a smooth map of the associated complex analytic spaces.

1.6. Sheaves on a topological space will be sheaves of Λ-modules for some fixed unital, commutative ring Λ of finite global dimension.

1.7. 'Local system' will mean 'locally constant sheaf' (not necessarily with finitely generated stalks).

1.8. For a topological space X, we write $D^+(X)$ for the bounded below derived category of sheaves. If X is a variety, $D(X)$ will denote the bounded derived category of algebraically constructible sheaves on the complex analytic space associated to X. Unless explicitly stated otherwise, functors on sheaves will always be derived. I.e., we write f_* instead of Rf_*, etc.

1.9. Given a topological group G, a G-torsor will mean a principal G-bundle. If G acts on a space X, we write $D^+_G(X)$ for the bounded below equivariant derived category of X (see [BL]). If G is an algebraic group acting (algebraically) on a variety X, then we write $D_G(X)$ for the bounded algebraically constructible equivariant derived category.

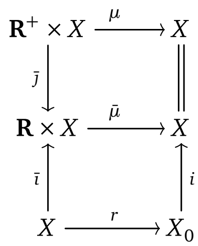
2. Conic sheaves

2.1. Let $\mathbf{R}^+$ denote the multiplicative group of strictly positive real numbers, and let X be a topological space endowed with an $\mathbf{R}^+$-action. As $\mathbf{R}^+$ is contractible, we may identify the $\mathbf{R}^+$-equivariant derived category $D^+_{\mathbf{R}^+}(X)$ with a full subcategory of $D^+(X)$. Objects in $D^+_{\mathbf{R}^+}(X)$ are called *conic*. An object $F \in D^+(X)$ is conic if and only if[1] for each orbit v of $\mathbf{R}^+$ in X, the cohomology sheaves $H^i(F)|_v$ are local systems for all $i \in \mathbf{Z}$ (see [BL, Proposition 3.7.3] or [KS, Proposition 3.7.2]).

Example 2.1.1. All local systems on X are conic.

[1] The 'only if' is immediate from equivariance. The other direction utilizes the contractibility of $\mathbf{R}^+$.

2.2. Let $i: X_0 \hookrightarrow X$ be a closed subspace. Write $\mu: \mathbf{R}^+ \times X \to X$ for the action map. We say that $\mathbf{R}^+$ *contracts* X to X_0 if the action μ extends to a map $\bar{\mu}: \mathbf{R} \times X \to X$ and there is a retraction $r: X \to X_0$ fitting into a commutative diagram



where $\bar{\imath}(x) = (0, x)$ and $\bar{\jmath}(t, x) = (t, x)$. We have the following variant of the homotopy invariance of cohomology:

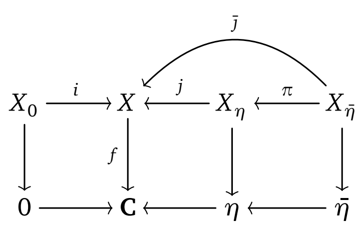
Proposition 2.2.1. *Let $F \in D^+_{\mathbf{R}^+}(X)$. Then i^* applied to the canonical map $r^* r_* F \to F$ yields an isomorphism $r_* F \xrightarrow{\cong} i^* F$.*

Proof. This is proved in [So, Proposition 1] with $\mathbf{R}^+$ (resp. $\mathbf{R}$) replaced by $\mathbf{C}^*$ (resp. $\mathbf{C}$). The argument[2] works here verbatim, with the aforementioned changes. Alternatively, a more hands on proof, for X a (real) vector bundle and X_0 its zero section, can be found in [KS, Proposition 3.7.5]. This proof also applies[3] to our general situation. □

Example 2.2.2. Take $X = \mathbf{C}$ and $f = \mathrm{id}$. Then $\mathbf{R}^+ \subset \mathbf{C}^*$ acts on X via multiplication and contracts it to $\{0\}$. Let L be a local system on $\mathbf{C} - \{0\}$. Then L is $\mathbf{R}^+$-equivariant. Write $j: \mathbf{C} - \{0\} \hookrightarrow \mathbf{C}$ for the inclusion. Now Proposition 2.2.1 recovers the familiar fact that the stalk cohomology of $j_* L$ at 0 is isomorphic to $H^*(\mathbf{C}^*; L)$.

3. Conic nearby cycles

3.1. Let $f: X \to \mathbf{C}$ be map of topological spaces. Given another map $S \to \mathbf{C}$, write X_S for the fibre product $X \times_{\mathbf{C}} S$. Let $\eta = \mathbf{C} - \{0\}$, and let $\bar{\eta} \to \eta$ be the universal cover of η. Then we have a commutative diagram, with all squares cartesian:



The nearby cycles functor $\psi_f: D^+(X_\eta) \to D^+(X_0)$ is defined by setting:

$$\psi_f(F) = i^* \bar{\jmath}_* \pi^*(F).$$

3.2. If $g: Y \to X$ is another map, base change yields a natural transformation $g^* \psi_f \to \psi_{fg} g^*$ (with an obvious abuse of notation for g^*). If g is smooth, then this is a canonical isomorphism. Similarly, we have a canonical map $\psi_f g_* \to g_* \psi_{fg}$ which is an isomorphism if g is proper.

[2] In turn extracted from [Sp] which treats the case $X_0 = *$

[3] However, I have not checked all the details.

3.3. Assume X and $\mathbf{C}$ are equipped with $\mathbf{C}^*$-actions and that f is an equivariant map. The evident group structure on $\eta = \mathbf{C}^*$ lifts (uniquely) to a group structure on $\bar{\eta}$ making $\bar{\eta} \to \eta$ a morphism of topological groups. This makes all spaces and maps in the diagram above $\bar{\eta}$-equivariant. Further, as $\bar{\eta}$ is contractible, the forgetful functor identifies the equivariant derived category $D^+_{\bar{\eta}}(X)$ with a full subcategory of $D^+(X)$.

Proposition 3.3.1. *Let $F \in D^+_{\bar{\eta}}(X_\eta)$. Then $\bar{j}_*\pi^*(F) \in D^+_{\mathbf{R}^+}(X)$, where $\mathbf{R}^+$ acts on X via the inclusion $\mathbf{R}^+ \subset \mathbf{C}^*$.*

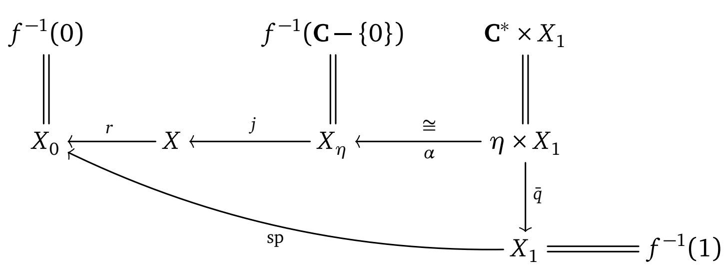
Proof. As all maps are $\bar{\eta}$-equivariant, $\bar{j}_*\pi^*(F) \in D^+_{\bar{\eta}}(X)$. Therefore, the cohomology sheaves of $\bar{j}_*\pi^*(F)$ are local systems when restricted to orbits of $\bar{\eta}$ in X. Since $\bar{\eta} \to \eta$ is surjective, $\mathbf{R}^+$-orbits in X are contained in $\bar{\eta}$-orbits. □

3.4. Now let $\mathbf{C}^*$ act on $\mathbf{C}$ via multiplication, and let X be a variety with a $\mathbf{C}^*$-action such that f is equivariant. Let $X_1 = f^{-1}(1)$. Then the map $\alpha\colon \mathbf{C}^* \times X_1 \to X_\eta$ defined by $\alpha(t,x) = t \cdot x$ is an equivariant isomorphism (with $\mathbf{C}^*$ acting on $\mathbf{C}^* \times X_1$ via $s \cdot (t,x) = (st,x)$). Let $\bar{q}\colon \eta \times X_1 \to X_1$ be the projection and set $q = \alpha^{-1}\bar{q}$

The notion of a contracting $\mathbf{C}^*$-action on a variety is defined exactly as in §2.2, but with $\mathbf{R}^+$ (resp. $\mathbf{R}$) replaced by $\mathbf{C}^*$ (resp. $\mathbf{C}$), and with all morphisms/diagrams required to lie in the category of varieties. Assume the $\mathbf{C}^*$-action on X contracts it to X_0. Let $r\colon X \to X_0$ be the retraction, and let $\bar{\mu}\colon \mathbf{C} \times X \to X$ be the extension of the action map that is part of the contraction data. Define $\mathrm{sp}\colon X_1 \to X_0$ by

$$\mathrm{sp}(x) = rj\alpha(1,x).$$

Lemma 3.4.1. *The following diagram commutes:*

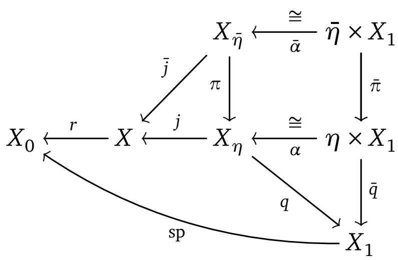


Proof. Let $(t,x) \in \eta \times X_1$. Then $rj\alpha(t,x) = \bar{\mu}(0, t \cdot x)$ and $\mathrm{sp}\bar{q}(t,x) = \bar{\mu}(0,x)$. For fixed x, each of these maps gives an extension of the morphism $\mathbf{C}^* \to X$, $t \mapsto t \cdot x$, to a map $\mathbf{C} \to X$. Since X is separated, any such extension is unique.[4] □

Proposition 3.4.2. *The functors $\psi_f q^*$ and sp_* are isomorphic.*

[4] In different language: the retraction $X \to X_0$ at a pont in X is given by taking the limit of its $\mathbf{C}^*$-orbit towards 0. As the complex analytic space associated to a variety is Hausdorff, this limit is uniquely defined.

Proof. Thanks to the Lemma, we have a commutative diagram (with top square cartesian):



Consequently:

$$
\begin{aligned}
\psi_f(q^*F) &= i^* \bar j_* \pi^* q^*(F) && \text{(by definition)}\\
&\cong r_* \bar j_* \pi^* q^*(F) && \text{(by Propositions 2.2.1and 3.3.1)}\\
&\cong r_* j_* \pi_* \pi^* q^*(F) && (\text{since } \bar j = j\pi)\\
&\cong r_* j_* (\alpha_* \bar\pi_* \bar\alpha_*^{-1})((\bar\alpha^{-1})^* \bar\pi^* \alpha^*) q^*(F) && (\text{as } \pi = \alpha\bar\pi\bar\alpha^{-1})\\
&\cong r_* j_* \alpha_* \bar\pi_* \bar\alpha_*^{-1} (\bar\alpha^{-1})^* \bar\pi^* \bar q^*(F) && (\text{as } \bar q = q\alpha)\\
&\cong r_* j_* \alpha_* \bar\pi_* \bar\pi^* \bar q^*(F) && (\text{since } (\bar\alpha^{-1})^* \cong \bar\alpha_*)\\
&\cong \mathrm{sp}_* \bar q_* \bar\pi_* \bar\pi^* q^*(F) && (\text{because } rj\alpha = \mathrm{sp}\bar q)\\
&\cong \mathrm{sp}_*(F) && (\text{since } \bar\eta \text{ is contractible}).
\end{aligned}
$$

□

4. Equivariant nearby cycles

4.1. Lifting ψ_f to equivariant derived categories requires no new ideas beyond those in [BL]: ψ_f commutes with smooth pullbacks (in turn a consequence of smooth base change). We use the notation of §3.1 and outline the main points.

4.2. Let X be a space endowed with an action of a topological group G. Assume $f : X \to \mathbf{C}$ is a G-*invariant* morphism. That is, $f(g \cdot x) = f(x)$, for all $g \in G$ and $x \in X$.

As in [BL], a *resolution* of X is a diagram

$$X \xleftarrow{p} X_G \xrightarrow{q} \bar X$$

with p equivariant and q a G-torsor. If p is smooth, then it is called a smooth resolution. We say that $\bar F \in D^+(\bar X)$ *comes from* $D^+(X)$ if there exists $F \in D^+(X)$ along with an isomorphism $p^*F \cong q^*\bar F$. The map f induces a morphism $\bar f : \bar X \to \mathbf{C}$. This yields the nearby cycles functor:

$$\psi_{\bar f} : D^+(\bar X_\eta) \to D(\bar X_0).$$

Lemma 4.2.1. *If the resolution is smooth and $\bar F \in D^+(\bar X_\eta)$ comes from $D^+(X_\eta)$, then $\psi_{\bar f}\bar F$ comes from $D^+(\bar X_0)$.*

Proof. By definition, there exists $F \in D^+(X_\eta)$ along with an isomorphism $p^*F \cong q^*\bar F$. The nearby cycles functor commutes with smooth pullback. Thus,

$$p^*\psi_f(F) \cong \psi_{fp}(p^*F) \cong \psi_{fp}(q^*\bar F) \cong q^*\psi_{\bar f}(\bar F).$$ □

Proposition 4.2.2. *If G is a Lie group, then the nearby cycles functor $D(X_\eta) \to D(X_0)$ lifts to the equivariant setting. That is, we have a functor, the* equivariant nearby cycles

$$\psi_f : D^+_G(X_\eta) \to D^+_G(X_0),$$

compatible with nearby cycles on $D(X_\eta)$, which satisfies all the usual properties of the ordinary nearby cycles functor.

Proof. Let $F \in D^+_G(X_\eta)$. This is the following collection of data: for each resolution $X_\eta \longleftarrow E \longrightarrow \bar{X}_\eta$, an object $\bar{F} \in D^+(\bar{X}_\eta)$ that comes from $D^+(X_\eta)$, along with compatibility of these objects under pullbacks. Lemma 4.2.1 yields an object $\psi_{\tilde{f}}(\bar{F}) \in D^+(\bar{X}_0)$, coming from $D^+(X_0)$, for each smooth resolution $X_0 \longleftarrow \tilde{E} \longrightarrow \bar{X}_0$. This family is compatible under smooth pullbacks. Since Lie groups admit sufficiently many acyclic smooth resolutions (in the sense of [BL, §3]), this family extends to one on all resolutions. This defines ψ_f with the requisite properties. □

References

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[Sp] T. A. Springer, *A purity result for fixed point varieties in flag manifolds*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **31** (1984).