ON THE GEOMETRIC HECKE ALGEBRA

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This is a report on work in progress [5]. We explain how several *a priori* disparate braid group actions on derived categories of sheaves are representations of the geometric Hecke algebra.

Let G be a connected reductive group over the complex numbers. Let $B \subseteq G$ be a Borel subgroup. The *geometric Hecke algebra*, denoted $\mathscr{H}$, is the B-equivariant derived category of mixed Hodge modules on the flag variety G/B. The *convolution* bifunctor $- \cdot -\colon \mathscr{H} \times \mathscr{H} \to \mathscr{H}$ is defined by the formula

$$M \cdot N = m_!(M \mathbin{\widetilde{\boxtimes}} N),$$

where $M \mathbin{\widetilde{\boxtimes}} N$ denotes the equivariant descent of $M \boxtimes N$ to $G \overset{B}{\times} G/B$, and m is the map induced by the multiplication on G. This is an associative operation and endows $\mathscr{H}$ with a monoidal structure.

The B-orbits in G/B (Schubert cells) are parameterized by the Weyl group W. For each $w \in W$, let $i_w\colon X_w \hookrightarrow G/B$ denote the inclusion of the orbit corresponding to w. Let $\ell\colon W \to \mathbf{Z}_{\geq 0}$ denote the length function. Set $\mathbf{T}_w = i_{w!}\underline{X_w}[\ell(w)]$, where $\underline{X_w}$ is the constant sheaf on X_w.

Convolution endows the Grothendieck group of $\mathscr{H}$ with a ring structure. It is folklore that this ring is isomorphic to the Hecke algebra of W. In fact,

$$\text{if } \ell(ww') = \ell(w) + \ell(w'), \text{ then } \mathbf{T}_w \cdot \mathbf{T}_{w'} = \mathbf{T}_{\mathbf{ww'}}.$$

In other words, the $\mathbf{T}_w$, $w \in W$, satisfy the braid relations (at the categorical as well as Grothendieck group levels). We go a bit further and show that the $\mathbf{T}_w$ are invertible objects under convolution (the unit is given by $\mathbf{T}_e$). The proof of invertibility is essentially an SL_2 computation combined with the Artin-Grothendieck vanishing theorem for affine morphisms.

An *$\mathscr{H}$-action* or *$\mathscr{H}$-representation* on a category $\mathcal{C}$ is a monoidal functor from $\mathscr{H}$ to the category of endofunctors of $\mathcal{C}$. As the $\mathbf{T}_w$ are invertible, $\mathscr{H}$-actions give actions of the braid group of W, via auto-equivalences, on $\mathcal{C}$. We now outline a menagerie of such $\mathscr{H}$-actions.

Let X be a variety with G-action. A small variation of the formula for convolution defines an $\mathscr{H}$-action on the B-equivariant derived category of X. The case when X is a spherical variety is of particular interest in representation theory.

The G-equivariant derived category of $G/B \times G/B$ is equivalent to $\mathscr{H}$. In terms of the former category, convolution can be described by the usual formalism of convolution of kernels. That is, by the formula

$$M \cdot N = p_{13*}(p_{12}^* M \otimes p_{23}^* N)[-\dim G/B],$$

where $p_{??}\colon G/B \times G/B \times G/B \to G/B \times G/B$ denotes projection on the named factors. Let X be a variety. A variation of the above formula gives an $\mathscr{H}$-action on the derived category of mixed Hodge modules on $G/B \times X$.

Forgetting mixed structures, we also obtain $\mathscr{H}$-actions on ordinary (equivariant and non-equivariant) derived categories of sheaves. Via the Riemann-Hilbert correspondence, these transfer to the setting of D-modules. In this way, we recover the intertwining functors of [1].

We now sketch how to obtain $\mathscr{H}$-actions on categories of *coherent* sheaves. The preliminary nature of some of these results forces us to be a bit fuzzy. Consequently, until a published version of [5] appears, the statements that follow should be treated with a dose of skepticism.

Part of the data underlying a mixed Hodge module on a smooth variety X is a D-module on X endowed with a good filtration (the Hodge filtration). Taking the associated graded of this filtered D-module defines a functor from mixed Hodge modules to $\mathbf{C}^*$-equivariant coherent sheaves on the cotangent bundle T^*X. Specializing to $X = G/B$, in which case $T^*(G/B) = \tilde{\mathcal{N}}$ (the enhanced nilpotent cone, à la the Springer resolution), this functor was exploited, at the level of Grothendieck groups, by T. Tanisaki [4]. We extend the results of [4] to the categorical level, and obtain an $\mathscr{H}$-action on $G \times \mathbf{C}^*$-equivariant coherent sheaves on $\tilde{\mathcal{N}}$. This extends the braid group actions of M. Khovanov and R. Thomas [2] to arbitrary type. Further, this action coincides (modulo minor 'twists') with the braid group actions constructed by R. Bezrukavnikov and S. Riche [3]. However, the results of [3] hold over arbitrary fields, ours are only over the complex numbers. This is perhaps indicative that our definition of $\mathscr{H}$ is itself just the shadow of a more fundamental motivic construct.

References

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