

$\text{PerMon}(X)$ - monoidal perverse sheaves on X (given $\mathbb{C}^* \curvearrowright X$)

All exactness statements w/ respect to perverse t-structure

$\text{PerMon}(pt) = k\text{-vector spaces}$

$\text{PerMon}(\mathbb{C}^*) = k\text{-vector spaces equipped w/ an automorphism } \tau$

$\text{PerMon}(\mathbb{C}) = \text{diagrams of vector spaces:}$

$$\Psi \xrightarrow{\text{can}} \Phi \xrightarrow{\text{val}} \Psi$$

such that $1 - \text{can} \cdot \text{val}$ (equivalently $1 - \text{val} \cdot \text{can}$) is invertible.

Equivalence given by:

$$M \mapsto \Psi(M)[-1] \xrightarrow{\text{can}} \Phi(M) \xrightarrow{\text{val}} \Psi(M)[-1]$$

Ψ, Φ are nearby, vanishing cycles functors, respectively.

~~Shift conventions are:~~ with respect to $\mathbb{C} \xrightarrow{\text{id}} \mathbb{C}$. Shift conventions are: $\Psi[-1]$ and Φ t-exact.

$$\{0\} \xleftarrow{i} \mathbb{C} \xleftarrow{i} \mathbb{C}^*$$

In terms of quivers:

i_* - exact.

$$V \mapsto (0 \rightarrow V \rightarrow 0)$$

i^* - right exact

$$H^0(i^*) : (\Psi \xrightarrow{\text{can}} \Phi \xrightarrow{\text{val}} \Psi) \mapsto \text{coker}(\text{can})$$

$$H^{-1}(i^*) : (\Psi \xrightarrow{\text{can}} \Phi \xrightarrow{\text{val}} \Psi) \mapsto \ker(\text{can})$$

$$\text{other } H^i(i^*) = 0$$

$i^!$ - left exact

$$H^0(i^!) : (\Psi \xrightarrow{\text{can}} \Phi \xrightarrow{\text{val}} \Psi) \mapsto \ker(\text{val})$$

$$H^1(i^!) : (\Psi \xrightarrow{\text{can}} \Phi \xrightarrow{\text{val}} \Psi) \mapsto \text{coker}(\text{val})$$

$$\text{other } H^i(i^!) = 0$$

j_* - exact

$$(V, T) \mapsto (V \xrightarrow{1-T} V \xrightarrow{\text{id}} V)$$

$j_!$ - exact

$$(V, T) \mapsto (V \xrightarrow{\text{id}} V \xrightarrow{1-T} V)$$

j^* - exact

$$(\mathcal{U} \xrightarrow{\text{can}} \mathcal{E} \xrightarrow{\text{val}} \mathcal{V}) \mapsto (\mathcal{U}, 1 - \text{val} \circ \text{can})$$

id - exact (derived duality)

$$(\mathcal{U} \xrightarrow{\text{can}} \mathcal{E} \xrightarrow{\text{val}} \mathcal{V}) \mapsto (\mathcal{U}^\vee \xrightarrow{\text{val}^\vee} \mathcal{E}^\vee \xrightarrow{\text{can}^\vee} \mathcal{V}^\vee)$$

$(-)^\vee = \text{vector space duality}$

$\psi[-1]$ - exact (nearby cycles)

$$(\mathcal{U} \rightarrow \mathcal{E} \rightarrow \mathcal{V}) \mapsto \mathcal{U}$$

ϕ - exact (vanishing cycles)

$$(\mathcal{U} \rightarrow \mathcal{E} \rightarrow \mathcal{V}) \mapsto \mathcal{E}$$

Fourier transform - exact (appropriately shifted) $\mathcal{F}$

$$\mathcal{F} : (\mathcal{U} \rightarrow \mathcal{E} \rightarrow \mathcal{V}) \mapsto (\mathcal{E} \rightarrow \mathcal{U} \rightarrow \mathcal{V})$$

sanity checks

skyscraper at 0, denoted $\mathcal{I}_0$, corresponds to

$$\mathcal{I}_* \mathcal{K} = 0 \rightarrow \mathcal{K} \rightarrow 0$$

constant sheaf on $\mathbb{C}$; (shifted to make perverse)

$$\frac{\mathcal{K}}{\mathbb{C}}[1] = \mathcal{K} \rightarrow 0 \rightarrow \mathcal{K}$$

$$\text{check } i^*(\mathcal{K} \rightarrow 0 \rightarrow \mathcal{K}) \text{ has } H^0 = \text{coker}(\mathcal{K} \rightarrow 0) = 0$$

$$H^{-1} = \ker(\mathcal{K} \rightarrow 0) = \mathcal{K}$$

$$\mathcal{F}(0 \rightarrow \mathcal{K} \rightarrow 0) = \mathcal{K} \rightarrow 0 \rightarrow \mathcal{K}$$

~~Proof~~ There are identifications:

$$\mathcal{F}(i_* \psi[-1]) = i_* \phi \mathcal{F}$$

$$\text{Proof } \mathcal{F}(i_* \psi[-1])(\mathcal{U} \xrightarrow{\text{can}} \mathcal{E} \xrightarrow{\text{val}} \mathcal{V}) = \mathcal{F}(i_*(\mathcal{U}))$$

$$= \mathcal{F}(0 \rightarrow \mathcal{U} \rightarrow 0)$$

$$= \mathcal{U} \rightarrow 0 \rightarrow \mathcal{U}$$

$$i_* \phi \mathcal{F}(\mathcal{U} \xrightarrow{\text{can}} \mathcal{E} \xrightarrow{\text{val}} \mathcal{V}) = i_* \phi(\mathcal{E} \rightarrow \mathcal{U} \rightarrow \mathcal{V})$$

$$= i_* \mathcal{U}$$

$$= \mathcal{U}$$

will be convenient to have one more functor:

$$\pi^+ : \text{PerMon}(\text{pt}) \longrightarrow \text{PerMon}(\mathbb{C})$$

$$\uparrow \text{ corresponds to } \pi^*[1]; \quad \begin{array}{c} \mathbb{C} \\ \downarrow \pi \\ \text{pt} \end{array}$$

π^+ - exact

$$v \longmapsto (v \rightarrow 0 \rightarrow v)$$

thm

We have a canonical isomorphism/identification:

$$\mathcal{F}(i_*\phi) \simeq \pi^+\psi(\mathcal{F})$$

proof

just by definition:

$$\begin{aligned} \mathcal{F}i_*\phi(\Psi \rightarrow \Phi \rightarrow \Psi) &= \mathcal{F}i_*\Phi \\ &= \mathcal{F}(0 \rightarrow \Phi \rightarrow 0) \\ &= (\Phi \rightarrow 0 \rightarrow \Phi) \end{aligned}$$

$$\begin{aligned} \pi^+\psi(\mathcal{F}(\Psi \rightarrow \Phi \rightarrow \Psi)) &= \pi^+\psi(\Phi \rightarrow \Psi \rightarrow \Phi) \\ &= \pi^+(\Phi) \\ &= \Phi \rightarrow 0 \rightarrow \Phi \end{aligned}$$

QED

complement:

Nothing really changes in the case of the trivial bundle

$$\begin{array}{c} X \times \mathbb{C} \\ \downarrow \\ X \end{array}$$

But this should imply the same

result for arbitrary vector bundles.

Not clear how to prove it purely topologically (ok w/ $\mathcal{D}$ -modules).