SOME GEOMETRIC FACETS OF THE LANGLANDS CORRESPONDENCE FOR REAL GROUPS

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1. **Introduction.** Let $G^\vee$ be a reductive algebraic group defined over $\mathbf{R}$. The local Langlands correspondence [L] describes the set of equivalence classes of irreducible admissible representations of $G^\vee(\mathbf{R})$ in terms of the Weil-Deligne group and the complex dual group G. Roughly, it partitions the set of irreducible representations into finite sets (L-packets) and then describes each L-packet. Since Langlands' original work, this has been refined in several directions. Most relevant to this document is the work of Adams-Barbasch-Vogan [ABV].

The key construction in [ABV] is that of a variety (the parameter space) on which G acts with finitely many orbits; each L-packet is re-interpreted as an orbit, and representations in the L-packet as the equivariant local systems supported on it. Adams-Barbasch-Vogan further demonstrate that the parameter space encodes significant character level information.

In [So] W. Soergel has outlined a conjectural relationship between the geometric and representation theoretic categories appearing in [ABV]. This relationship, roughly a type of Koszul duality, yields a conceptual explanation for the phenomena observed in [ABV].[1] The current note was born in an attempt to settle Conjectures 4.2.2, 4.2.3, 4.2.6 in [So], and Soergel's 'Equivariant Formality' conjecture (implicit in [So]; see §6 and [Lun, §0.2]). These conjectures describe the structure of the geometric categories appearing in [ABV]; the current document has little to say about Soergel's Basic Conjecture (relating the geometric categories to representation theory).

We '*almost*' succeed (see §6). Soergel's conjectures are formulated in graded versions of our categories (in the sense of [So, §4]). These 'graded representation theories' are not constructed here because of a rather frustrating reason: we use the language of Hodge modules, and the category of Hodge modules is too large for the purposes of graded representation theory (see §6 and footnote 3).

The main result is Theorem 4.4 describing the Hodge structure on equivariant $\mathrm{Ext}^\bullet$ between simple perverse sheaves on the parameter space.[2] It yields a host of ancillary results which are of independent interest: Corollary 4.5 and 4.6 ('parity vanishing'); Theorem 5.5 and Corollary 5.6 ('positivity' of a Hecke algebra module). An informal discussion regarding Soergel's graded categories is contained in §6.

[1] The localization theorem of Beilinson-Bernstein also establishes a relationship between representation theory and geometry. However, Soergel's approach is very different: localization leads to geometry on the group itself; Soergel's approach results in geometry on the dual group.

[2] The parameter space is never explicitly mentioned. The translation between the symmetric varieties G/K appearing below and the parameter space is provided by [ABV, Proposition 6.24]. The parameter space is essentially a disjoint union of varieties of the form G/K.

Acknowledgments: This work was conceived while I was visiting the Albert-Ludwigs-Universität, Freiburg, in the summer of 2012. I am indebted to W. Soergel for his hospitality, his continuing explanations and patience.[3] I am further grateful to W. Soergel and M. Wendt for sharing their beautiful ideas on the use of motivic sheaves in representation theory.

2. **Conventions.** Throughout, 'variety' = 'separated reduced scheme of finite type over $\mathrm{Spec}(\mathbf{C})$'. A *fibration* will mean a morphism of varieties which is locally trivial (on the base) in the ètale sense. Constructible sheaves, cohomology, etc. will always be with $\mathbf{R}$ or $\mathbf{C}$ coefficients, and with respect to the complex analytic site associated to a variety.

Given an algebraic group G, we write G^0 for its identity component. Suppose G acts on X. Then we write G_x for the isotropy group of a point $x \in X$. Given a principle G-fibration $E \to B$, we write $E \times^G X \to B$ for the associated fibration.[4]

We write $D_G(X)$ for the G-equivariant derived category (in the sense of [BL]), and $\mathrm{Perv}_G(X) \subseteq D_G(X)$ for the abelian subcategory of equivariant perverse sheaves on X. Perverse cohomology is denoted by ${}^pH^*$. Change of group functors (restriction, induction equivalence, quotient equivalence, etc.) will often be omitted from the notation. All functors between derived categories will be tacitly derived. Both the functor of G-equivariant cohomology as well as the G-equivariant cohomology ring of a point will be denoted by H^*_G.

3. $\mathbf{B}\backslash\mathbf{G}/\mathbf{K}$. Let G be a connected reductive group, $\theta\colon G \to G$ a non-trivial algebraic involution, T a θ-stable maximal torus, and $B \supseteq T$ a θ-stable Borel containing it (such a pair (B,T) always exists, see [St, §7]). Write W for the Weyl group. Let $K = G^\theta$ denote the fixed point subgroup. Then

(i) K is reductive (but not necessarily connected, see [V, §1]);
(ii) $|B\backslash G/K| < \infty$ (a convenient reference is [MS, §6]);
(iii) K-orbits in G/B are affinely embedded (see [M, Ch. H, Proposition 1]);
(iv) for each $x \in G/B$, the component group K_x/K_x^0 has exponent 2 [V, Proposition 7].

Our primary concern is the category $D_{B\times K}(G)$, for the $B \times K$-action given by $(b,k)\cdot g = bgk^{-1}$. The evident identification of $B \times K$-orbits in G, with B-orbits in G/K, and with K-orbits in G/B, respects closure relations. There are corresponding identifications: $D_B(G/K) = D_{B\times K}(G) = D_K(G/B)$. These identifications will be used without further comment.

Let $s \in W$ be a simple reflection, $P \supseteq B$ the corresponding minimal parabolic, and v a B-orbit in G/K. Then the subvariety $P \cdot v \subseteq G/K$ contains a unique open dense B-orbit $s \star v$. Let $\leq$ denote the closure order on orbits, i.e., $v \leq w$ if and only if v is contained in the closure $\overline{w}$.

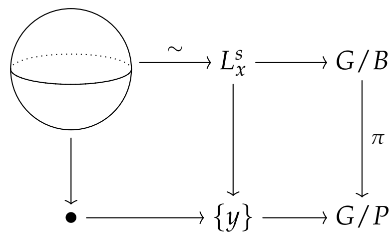
[3] In January, 2010, W. Soergel explained to me: *"In a way, there should be a better category than what we work with, sort of much more motivic, where these problems disappear. Think about Grothendieck's conjecture: the action of Frobenius on the ètale cohomology of a smooth projective variety should be semisimple! So this non-semisimplicity is sort of due to the fact we are not working with motives, but with some rather bad approximation, I suggest."* At my glacial pace it has taken me four years to appreciate this (see §6).

[4] Generally, $E \times^G X$ is only an algebraic space. It is a variety if, for instance, X is quasi-projective with linearized G-action; or G is connected and X can be equivariantly embedded in a normal variety (Sumihiro's Theorem). One of these assumptions will always be satisfied below.

Theorem 3.1 ([RS, Theorem 4.6]). *If $w \in B\backslash G/K$ is not closed, then there exists a simple reflection $s \in W$, and $v \in B\backslash G/K$ such that $v \lneq w$ and $s \star v = w$.*

Let $\pi\colon G/B \to G/P$ be the evident projection. Let $x \in G/B$. Set $y = \pi(x)$, and $L_x^s = \pi^{-1}(y)$. Note: $L_x^s \simeq \mathbf{P}^1$.

(∗)

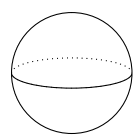


The K-action induces an isomorphism $K \times^{K_y} L_x^s \xrightarrow{\sim} K \cdot L_x^s$. Thus,

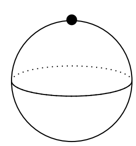
$$D_K(K \cdot L_x^s) = D_K(K \times^{K_y} L_x^s) = D_{K_y}(L_x^s) = D_{K_y}(\mathbf{P}^1).^{5}$$

As $|B\backslash G/K| < \infty$, the image of K_y in $\mathrm{Aut}(L_x^s)$ has dimension ≥ 1. Identify $\mathbf{P}^1$ with $\mathbf{C} \sqcup \{\infty\}$. Modulo conjugation by an element of $\mathrm{Aut}(L_x^s) \simeq PGL_2$, there are four possibilities for the decomposition of $\mathbf{P}^1$ into K_y-orbits:

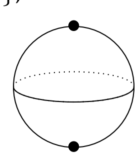
Case G: $\mathbf{P}^1$ (the action is transitive);



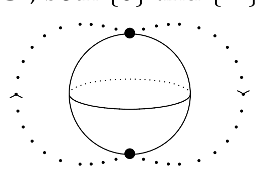
Case U: $\mathbf{P}^1 = \mathbf{C} \sqcup \{\infty\}$;



Case T: $\mathbf{P}^1 = \{0\} \sqcup \mathbf{C}^* \sqcup \{\infty\}$;



Case N: $\mathbf{P}^1 = \{0, \infty\} \sqcup \mathbf{C}^*$; both $\{0\}$ and $\{\infty\}$ are fixed points of K_y^0.



We will say that w is of type **G**, **U**, **T** or **N** relative to s depending on which of these decompositions actually occurs.

Given an irreducible equivariant local system V_τ on a K-orbit $j\colon w \hookrightarrow G/B$, set

$$\mathcal{L}_\tau = j_{!*} V_\tau[d_\tau], \quad \text{where} \quad d_\tau = \dim(w).$$

[5] This is the analogue of the Lie theoretic principle that 'local phenomena is controlled by SL_2'.

Call $\mathcal{L}_\tau$ *clean* if $\mathcal{L}_\tau \simeq j_! V_\tau[d_\tau]$. Call $\mathcal{L}_\tau$ *cuspidal* if for each simple reflection s, each $v \neq w$ with $s \star v = w$, and each K-equivariant local system V_γ on v, the object $\mathcal{L}_\tau$ does *not* occur as a direct summand of ${}^pH^*(\pi^*\pi_*\mathcal{L}_\gamma)$, where π is as in ($*$).[6]

Lemma 3.2 ([MS, Lemma 7.4.1])**.** *Cuspidals are clean.*

Proof. As indicated, this is [MS, Lemma 7.4.1]. Regardless, the language employed in [MS] is a bit different from ours, so we sketch the argument in order to orient the reader.

Let $j\colon w \hookrightarrow G/B$ be a K-orbit, and V_τ a local system on w such that $\mathcal{L}_\tau$ is cuspidal. Write $\overline{w}$ for the closure of w. To demonstrate the assertion we need to show that $(j_*V_\tau)|_v = 0$ for each orbit v in $\overline{w} - w$.

If s is a simple reflection such that $s \star w = w$ and $P_s \cdot w$ contains an orbit other than w, then as $\mathcal{L}_\tau$ is cuspidal, w must be of type **T** or **N** relative to s. In the language of [MS], this means that each such s is 'of type IIIb or IVb for w'. Let I be the set consisting of simple reflections s as above and let P_I be the parabolic subgroup of G containing B and corresponding to I. Then in [MS, §7.2.1] it is shown that $P_I \cdot w = \overline{w}$.

Now if v is an orbit of codimension 1 in $\overline{w}$, then there exists $s \in I$ such that $s \star v = w$. Inspecting the cases **T** and **N** yields the required vanishing in this case.

For arbitrary v, proceed by induction on codimension. Let $s \in I$ be such that $s \star v > v$. Let π be as in ($*$). As $\mathcal{L}_\tau$ is cuspidal, $\pi_*(j_*V_\tau) = 0$. Furthermore, if $(j_*V_\tau)|_v \neq 0$, then $\pi_*((j_*V_\tau)|_v) \neq 0$. All of this follows by inspection of the cases **G**, **U**, **T**, **N**. Combined these vanishing and non-vanishing statements imply $(j_*V_\tau)|_v \neq 0$ only if $(j_*V_\tau)|_{s\star v} \neq 0$. Thus, applying the induction hypothesis yields the result. □

4. **Mixed structures.** Given a variety X, write $M(X)$ for the category of **R**-mixed Hodge modules on X, and $DM(X)$ for its bounded derived category [Sa]. If a linear algebraic group acts on X, write $DM_G(X)$ for the corresponding mixed equivariant derived category. When dealing with mixed as well as ordinary categories, objects in mixed categories will be adorned with an ^{H}. Omission of the ^{H} will denote the classical object underlying the mixed structure.

A mixed Hodge structure is called *Tate* if it is a successive extension of Hodge structures of type (n, n). A mixed Hodge module $\mathcal{A}^H \in M(X)$ will be called $*$*-pointwise Tate* if, for each point $i\colon \{x\} \hookrightarrow X$, the stalk $H^*(i^*\mathcal{A}^H)$ is Tate. Call $\mathcal{A}^H \in DM(X)$ $*$-pointwise Tate if each ${}^pH^i(\mathcal{A}^H)$ is so. An object of $DM_G(X)$ is $*$-pointwise Tate if it is so under the forgetful functor $DM_G(X) \to DM(X)$.

Lemma 4.1. *Let π be as in ($*$). Then $\pi^*\pi_*$ preserves the class of $*$-pointwise Tate objects.*

Proof. Use the notation surrounding ($*$). Then the assertion reduces to the claim that if $\mathcal{A}^H \in M_{K_y}(L_x^s)$ is $*$-pointwise Tate, then $H^*(L_x^s; \mathcal{A}^H)$ is Tate.[7] This is immediate from the possible K_y-orbit decompositions **G**, **U**, **T** and **N**. □

[6] The term 'cuspidal' has a very specific meaning in representation theory. It is not clear to me whether the terminology is completely justified in the current geometric setting.

[7] The core of this argument is due to R. MacPherson (in a 'parity vanishing' context for Schubert varieties), see [So2000, Lemma 3.2.3].

Each irreducible $B \times K$-equivariant local system V_τ, on an orbit w, underlies a unique (up to isomorphism) polarizable variation of Hodge structure of weight zero. Denote this variation by V_τ^H. Taking intermediate extension, we obtain a pure (equivariant) Hodge module $\mathcal{L}_\tau^H$ of weight $d_\tau = \dim(w)$, i.e.,

$$\mathcal{L}_\tau^H = j_{!*} V_\tau^H[d_\tau],$$

where $j\colon w \hookrightarrow G$ is the inclusion.

Proposition 4.2. *$\mathcal{L}_\tau^H$ is $*$-pointwise Tate.*

Proof. Work in G/B. The statement is true for cuspidals, since they are clean (Lemma 3.2). The general case follows by induction (employing Theorem 3.1) and Lemma 4.1. □

Proposition 4.3. *Let $i\colon v \hookrightarrow G$ be the inclusion of a $B \times K$-orbit. Then $i^* \mathcal{L}_\tau^H$ is pure.*

Proof. Work in G/K. According to [MS, §6.4] (also see the comments at the end of §1 in [LV]), each B-orbit admits a contracting slice in the sense of [MS, §2.3.2]. This implies purity (see [MS, §2.3.2] or [KL, Lemma 4.5] or [So89, Proposition 1]). □

Given an algebraic group L acting on a variety X, set

$$\mathrm{Ext}_L^i(-,-) = \mathrm{Hom}_{D_L(X)}(-,-[i]).$$

The Hodge modules $\mathcal{L}_v^H$ endow each $\mathrm{Ext}^\bullet_{B\times K}(\mathcal{L}_\tau, \mathcal{L}_\gamma)$ with a Hodge structure.

Theorem 4.4. *$\mathrm{Ext}^\bullet_{B\times K}(\mathcal{L}_\tau, \mathcal{L}_\gamma)$ is Tate and pure of weight $d_\gamma - d_\tau$.*[8]

Proof. Work in G/K. Filtering $\mathrm{Ext}^\bullet_B(\mathcal{L}_\tau, \mathcal{L}_\gamma)$ by the orbit stratification, one sees that it suffices to argue that $\mathrm{Ext}^\bullet_B(i^*\mathcal{L}_\tau, i^!\mathcal{L}_\gamma))$ is pure and Tate for each B-orbit inclusion $i\colon u \hookrightarrow G/K$. Proposition 4.2 and Proposition 4.3 imply that both $i^*\mathcal{L}_\tau^H$ and $i^!\mathcal{L}_\gamma^H$ are direct sums of (shifted) one dimensional variations of Hodge structure. As H_L^* is pure and Tate, for any linear algebraic group L (cf. [D, §9.1]), the assertion is immediate. □

Corollary 4.5. $\mathrm{Ext}^i_{B\times K}(\mathcal{L}_\tau, \mathcal{L}_\gamma) = 0$ *unless* $i = d_\tau + d_\gamma \mod 2$.

Corollary 4.6. *$H^*_{B\times K}(G; \mathcal{L}_\tau)$ vanishes in either all even or all odd degrees.*

Some remarks are in order:

(i) Corollary 4.5 should be compared with [So, Conjecture 4.2.6]; also see §6.
(ii) Corollary 4.6 is essentially contained in [LV]. Lusztig-Vogan work in the non-equivariant ℓ-adic setting, but the Hecke algebra computations in [LV] can be used to obtain Corollary 4.6; also see §5. Note that Lusztig-Vogan rely on explicit calculations with the Hecke algebra and arguments from representation theory. An argument analogous to [LV], involving Hecke algebra computations (but no representation theory), can be found in [MS].
(iii) Let $P \supseteq B$ be a parabolic subgroup. One should be able to obtain similar results for the analogous $P \times K$-action on G using the technique of [So89].

[8] **Warning:** the non-equivariant analogue of this result is false!

5. **Hecke algebra.** Let $L \subseteq G$ be a closed subgroup (we are mainly interested in $L = B$ or K). Let $B \times L$ act on G via $(b, l) \cdot g = bgl^{-1}$. Define a bifunctor

$$- \star -: DM_{B\times B}(G) \times DM_{B\times L}(G) \to DM_{B\times L}(G),$$

called *convolution*, as follows. For $M \in DM_{B\times B}(G)$, $N \in DM_{B\times L}(G)$, the object $M \boxtimes N$ descends to an object $M \widetilde{\boxtimes} N \in DM_{B\times L}(G \times^B G)$. Set $M \star N = m_!(M \widetilde{\boxtimes} N)$, where $m\colon G \times^B G \to G$ is the map induced by multiplication. This operation is associative in the evident sense. As m is projective, convolution adds weights and commutes with Verdier duality (up to shift and Tate twist).

Taking $L = B$ yields a monoidal structure on $DM_{B\times B}(G)$. For each $w \in W$, set

$$\mathbf{T}_w = j_{w!}\underline{BwB} \quad \text{and} \quad \mathbf{C}_w = (j_{w!*}\underline{BwB}[\dim(BwB)])[-\dim(BwB)],$$

where $j_w\colon BwB \hookrightarrow G$ is the inclusion, and $\underline{BwB}$ denotes the trivial (weight 0) variation of Hodge structure on BwB. The unit for convolution is $\mathbf{1} = \mathbf{T}_e$.

Proposition 5.1. *The $\mathbf{T}_w$ satisfy the braid relations. That is, if $\ell(vw) = \ell(v) + \ell(w)$, then $\mathbf{T}_v \star \mathbf{T}_w = \mathbf{T}_{vw}$, where $\ell\colon W \to \mathbf{Z}_{\geq 0}$ is the length function.*

Proof. Multiplication yields an isomorphism $BvB \times^B BwB \xrightarrow{\sim} BvwB$. □

Proposition 5.2. *Let $s \in W$ be a simple reflection, and let π be as in (∗). Then, under the equivalence $DM_{B\times K}(G) \xrightarrow{\sim} DM_K(G/B)$, convolution with $\mathbf{C}_s$ is identified with $\pi^*\pi_*$.*

Proof. Left to the reader (see [So2000, Lemma 3.2.1]). □

Let $\mathbf{H}_q \subseteq DM_{B\times B}(G)$ be the triangulated subcategory generated by the $\mathbf{C}_w$, $w \in W$, and Tate twists thereof.

Proposition 5.3. *$\mathbf{H}_q$ is stable under convolution.*

Proof. This follows from [So, Proposition 3.4.6]. Alternatively, note

$$DM_{B\times B}(G) \xrightarrow{\sim} DM_B(G/B) \xrightarrow{\sim} DM_G(G \times^B G/B) \xrightarrow{\sim} DM_G(G/B \times G/B).$$

This puts us in the setting of the previous sections (with group $G \times G$ and involution $\theta(g_1, g_2) = (g_2, g_1)$). Now use Lemma 4.1 and Proposition 5.2. □

Corollary 5.4. *Each $\mathbf{T}_w$ is in $\mathbf{H}_q$.*

Proof. In view of Proposition 5.1, it suffices to prove this for each simple reflection s. In this case we have a distinguished triangle $\mathbf{1}[-1] \to \mathbf{T}_s \to \mathbf{C}_s \rightsquigarrow$ □

Let $\mathbf{M}_q \subseteq DM_{B\times K}(G)$ be the triangulated subcategory generated by the $\mathcal{L}_\tau^H$ and Tate twists thereof.

Theorem 5.5. *$\mathbf{M}_q$ is stable under convolution with objects of $\mathbf{H}_q$.*

Proof. Combine Lemma 4.1 with Proposition 5.2. □

Let $H_q = K_0(\mathbf{H}_q)$ and $M_q = K_0(\mathbf{M}_q)$ be the respective Grothendieck groups. These are free $\mathbf{Z}[q^{\pm 1}]$-modules via $q[\mathcal{A}] = [\mathcal{A}(-1)]$, where (-1) is the inverse of Tate twist. Convolution makes H_q a $\mathbf{Z}[q^{\pm 1}]$-algebra, and M_q an H_q-module.

Corollary 5.6. *The coefficients $c^{w}_{\gamma,\tau}(q)$ in the expansion*

$$(-1)^{d_\tau}[\mathbf{C}_w \star \mathcal{L}^H_\tau] = \sum_\gamma (-1)^{d_\gamma} c^w_{\gamma,\tau}(q)[\mathcal{L}^H_\gamma],$$

are polynomials in $q^{\pm 1}$ with non-negative coefficients.

The algebra H_q is isomorphic to the Iwahori-Hecke algebra associated to W. That is, H_q is isomorphic to the $\mathbf{Z}[q^{\pm 1}]$ algebra on generators T_w, $w \in W$, with relations: $T_v T_w = T_{vw}$ if $\ell(vw) = \ell(v) + \ell(w)$; and $(T_s + 1)(T_s - q) = 0$ if $\ell(s) = 1$. The isomorphism is given by $T_w \mapsto [\mathbf{T}_w]$. The convolution product $\mathbf{C}_w \star \mathcal{L}^H_\tau$, the groups $H^*_{B\times K}(G; \mathcal{L}_\tau)$, $\mathrm{Ext}^\bullet(\mathcal{L}_\tau, \mathcal{L}_\gamma)$, $\mathrm{Ext}^\bullet(j_! V_\tau, \mathcal{L}_\gamma)$, can all be explicitly computed via the module M_q: $\mathbf{C}_w \star \mathcal{L}^H_\tau$ because of Theorem 5.6; the rest because they are pure and Tate (Theorem 4.4) and can consequently be recovered from their weight polynomials. For an explicit description of M_q, see [LV] and [MS].

Corollary 5.6 appears to be new (although it might be possible to deduce it from the results of [LV]). It is a generalization of the well known positivity result for the Kazhdan-Lusztig basis (the classes $[\mathbf{C}_w]$) in the Hecke algebra.

6. **Informal remarks.** Let $\mathcal{A}$ be an abelian category, $\mathrm{Ho}(\mathcal{A})$ the homotopy category of chain complexes in $\mathcal{A}$, and $D(\mathcal{A})$ the derived category of $\mathcal{A}$. Given a collection of (bounded below) complexes $\{T_i\}$ each of whose components are injectives, set $T = \bigoplus_i T_i$. The complex $\mathcal{E} = \mathrm{End}^\bullet_{\mathcal{A}}(T)$ has an evident dg-algebra structure. Let $e_i \in \mathcal{E}$ denote the idempotent corresponding to projection on T_i. The functor $\mathrm{Hom}^\bullet(\mathcal{E}, -)$ yields an equivalence between the full triangulated subcategory of $D(\mathcal{A})$ generated by the T_i and the full triangulated subcategory of the dg-derived category $\mathrm{dgDer}-\mathcal{E}$ (of right dg $\mathcal{E}$-modules) generated by the $e_i\mathcal{E}$.

The dg-algebra $\mathcal{E}$, and hence $D(\mathcal{A})$, becomes significantly more tractable if $\mathcal{E}$ is *formal*, i.e., quasi-isomorphic to its cohomology $H^*(\mathcal{E})$ (viewed as a dg-algebra with trivial differential). In general, it can be difficult to establish formality. However, there is a criterion due to P. Deligne: if $\mathcal{E}$ is endowed with an additional $\mathbf{Z}$-grading $\mathcal{E}^i = \bigoplus_{j\in\mathbf{Z}} \mathcal{E}^{i,j}$ which is respected by the differential, and each $H^i(\mathcal{E})$ is concentrated in degree i (for the additional grading), then $\mathcal{E}$ is formal.

In the setting of the previous sections, let $\mathcal{L} = \bigoplus_\tau \mathcal{L}_\tau$ be the direct sum of the simple objects in $\mathrm{Perv}_{B\times K}(G)$. Let $\mathcal{E} = \mathrm{Ext}^\bullet_{B\times K}(\mathcal{L}, \mathcal{L})$, viewed as a dg-algebra with trivial differential. Assume that the category $\mathbf{M}_q$ of the previous section is the derived category of an abelian category containing enough injectives. Further, assume that the forgetful functor $\mathbf{M}_q \to D_{B\times K}(G)$ yields a grading (via the weight filtration) in the sense of [BGS, §4]. Then, modulo some finiteness adjectives, Theorem 4.4 and Deligne's criterion yield $D_{B\times K}(G) \simeq \mathrm{dgDer}-\mathcal{E}$ (this is Soergel's Formality Conjecture). Conjectures 4.2.2, 4.2.3 and 4.2.6 of [So] also follow.

Now $D_{B\times K}(G)$ is not the derived category of an abelian category, but this is not a serious problem for implementing the above argument. However, $\mathbf{M}_q \to D_{B\times K}(G)$ simply does not yield a grading. The category of Hodge modules is too large. The issue is already visible over a point, since the category of Tate mixed Hodge structures is larger than the category of graded vector spaces. Further, isolating a suitable subcategory of $\mathbf{M}_q$ seems to be quite difficult (cf. [BGS, §4.5]). W. Soergel has explained to me how combining the arguments of this note with a joint project of his and M. Wendt's on 'motivic representation theory' (see [SW]) should allow this basic idea to be carried through (also see footnote 3). This perspective is also explicit in [BGS, §4] and [B, §G].

References

[ABV] J. Adams, D. Barbasch, D.A. Vogan, Jr., *The Langlands classification and irreducible characters for real reductive groups*, Progress in Mathematics **104**, Birkhäuser (1992).

[B] A. Beilinson, *Localization of representations of reductive Lie algebras*, Proc. ICM (1983), Warszawa.

[BGS] A. Beilinson, V. Ginzburg, W. Soergel, *Koszul duality patterns in Representation Theory*, J.A.M.S. **9** (1996), 473-526.

[BL] J. Bernstein, V. Lunts, *Equivariant sheaves and functors*, Lecture Notes in Math. **1578**, Springer-Verlag, Berlin (1994).

[D] P. Deligne, *Théorie de Hodge III*, Pub I.H.É.S **44** (1974), 5-77.

[L] R.P. Langlands, *On the Classification of Irreducible Representations of Real Algebraic Groups*, preprint I.A.S. (1973), appeared in Math. Surveys and Monographs **31**, AMS (1988).

[Lun] V. Lunts, *Equivariant sheaves on toric varieties*, Compositio Math. **96**, no. 1 (1995), 63-83.

[KL] G. Lusztig, D. Kazhdan, *Schubert varieties and Poincaré duality*, in Geometry of the Laplace operator, Proc. of Symposia in Pure Math XXXVI, AMS (1980).

[LV] G. Lusztig, D.A. Vogan, Jr., *Singularities of closures of K-orbits in flag manifolds*, Invent. Math. **71** (1983), 365-379.

[MS] J.G.M. Mars, T.A. Springer, *Hecke algebra representations related to spherical varieties*, Rep. Theory **2** (1998), 33-69.

[M] D. Miličić, *Localization and Representation Theory of Reductive Lie Groups*, book draft, available at http://www.math.utah.edu/~milicic/.

[RS] R.W. Richardson, T.A. Springer, *The Bruhat order on symmetric varieties*, Geom. Dedicata **35** (1990), 389-436.

[Sa] M. Saito, *Mixed Hodge modules*, Publ. RIMS **26** (1990), 221-333.

[So89] W. Soergel, *$\mathfrak{n}$-cohomology of simple highest weight modules on walls and purity*, Invent. Math. **98** (1989), 565-580.

[So2000] W. Soergel, *On the relation between intersection cohomology and representation theory in positive characteristic*, J. Pure and Applied Algebra **152** (2000), 311-335.

[So] W. Soergel, *Langlands philosophy and Koszul duality*, in Proc. NATO ASI 2000 in Constanta, Algebra-Representation Theory, Kluwer (2001), 379-414.

[SW] W. Soergel, M. Wendt, *Perverse motives and graded derived category $\mathcal{O}$*, arXiv:1404.6333v1.

[St] R. Steinberg, *Endomorphisms of linear algebraic groups*, Mem. A.M.S. **80** (1968).

[V] T. Vust, *Opération de groupes réductifs dans un type de cônes presque homogènes*, Bull. Math. Soc. France **102** (1974), 317-334.