A REMARK ON BRAID GROUP ACTIONS ON COHERENT SHEAVES

R. VIRK

1. INTRODUCTION

Examples of braid group actions on derived categories of coherent sheaves are abundant in the literature. Much of the interest in these stems from the relation with homological mirror symmetry, see [ST]. The purpose of this note is to give a construction of braid group actions on coherent sheaves (algebraic) via actions on derived categories of *constructible sheaves* (topological). The bridge between these worlds is provided by M. Saito's theory of mixed Hodge modules.

In §3 we review the construction of the main player: $D_m^b(B \backslash G/B)$, the Borel equivariant derived category of mixed Hodge modules on the flag variety G/B associated to a reductive group G. The key points are Prop. 3.3 (braid relations) and Thm. 3.8 (invertibility of the objects giving the braid relations). The contents of this section can be found in various forms in the literature, for instance see [Sp82].

Underlying a mixed Hodge module M on a smooth variety X is a filtered $\mathcal{D}$-module. The associated graded is a $\mathbf{C}^*$-equivariant coherent sheaf $\tilde{g}rM$ on the cotangent bundle T^*X. This brings us to the main result, Thm. 4.8, which exploits $\tilde{g}r$ to obtain a monoidal functor from $D_m^b(B \backslash G/B)$ to an appropriate category of coherent sheaves $\mathscr{H}$ on the Steinberg variety. In view of Prop. 3.3 and Thm. 3.8, this realizes our goal. Via standard Fourier-Mukai formalism, $\mathscr{H}$ acts on auxilliary categories of coherent sheaves giving braid group actions on these too.

The idea to exploit $\tilde{g}r$ in this fashion comes from T. Tanisaki's beautiful paper [Ta]. This theme was also explored by I. Grojnowski [Groj]. I. Grojnowski and T. Tanisaki work at the level of Grothendieck groups, we insist on working at the categorical level. Regardless, all the key ideas are contained in [Ta]. Furthermore, the key technical result (Thm. 4.5) that is used to prove Thm. 4.8 is due to G. Laumon [La2].

A variant of Thm. 4.8 has also been obtained by R. Bezrukavnikov and S. Riche [BR2]. Further, R. Bezrukavnikov and S. Riche were certainly aware of such a result long before this note was born (see [BR1]). Thus, experts in geometric representation theory have known that such a result must hold for a long time. Certainly V. Ginzburg (see [G1]) I. Grojnowski (see [Groj]), M. Kashiwara (see

[KaT]), D. Kazhdan (see [KL]), G. Lusztig (see [KL]), R. Rouquier (see [R]), and of course T. Tanisaki (see [Ta]) must have also known. Undoubtedly this list is woefully incomplete.

2. Conventions

Throughout 'variety' = 'separated reduced scheme of finite type over $\mathrm{Spec}(\mathbf{C})$'. A variety can and will be identified with its set of geometric points. If X is a variety, set $d_X = \dim_{\mathbf{C}}(X)$. If Y is another variety, set $d_{X/Y} = d_X - d_Y$.

Write $\mathrm{MHM}(X)$ for the abelian category of mixed Hodge modules on X, and $D_m^b(X)$ for its bounded derived category. The constant (mixed Hodge) sheaf is denoted $\underline{X}$. Tate twist is denoted by (1).

Let G be a linear algebraic group acting on X (action will always mean left action). Write $D_m^b(G\backslash X)$ for the G-equivariant derived category (in the sense of [BL]) of mixed Hodge modules on X. Write $\mathrm{For}\colon D_m^b(G\backslash X) \to D_m^b(X)$ for the forgetful functor.

We write $\mathcal{O}_X$ for the structure sheaf of X, $D^b(\mathcal{O}_X)$ for the (bounded) derived category of $\mathcal{O}_X$-coherent sheaves and $\mathrm{Coh}(\mathcal{O}_X) \subseteq D^b(\mathcal{O}_X)$ for the abelian subcategory of coherent sheaves. If an algebraic group G acts on X, we write $D^G(\mathcal{O}_X)$ for the (bounded) derived category of G-equivariant $\mathcal{O}_X$-coherent sheaves and $\mathrm{Coh}^G(\mathcal{O}_X) \subseteq D^G(\mathcal{O}_X)$ for the abelian subcategory of G-equivariant coherent sheaves.

Let $f\colon X \to Y$ be a morphism of varieties. For coherent sheaves, write f^{-1} for the ordinary pullback of sheaves, so that the pullback $f^*\colon D^b(\mathcal{O}_Y) \to D^b(\mathcal{O}_X)$ is given by $f^*M = \mathcal{O}_Y \otimes^L_{f^{-1}\mathcal{O}_X} f^{-1}M$. If X is smooth, write Ω_X for the cotangent sheaf and set $\omega_X = \bigwedge^{d_X} \Omega_X$. If $f\colon X \to Y$ is a morphism between smooth varieties, set $\omega_{X/Y} = \omega_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\omega_Y^{-1}$.

Write $\mathcal{D}_X$ for the sheaf of differential operators on X. A $\mathcal{D}_X$-module will always mean a coherent left $\mathcal{D}_X$-module which is quasi-coherent as an $\mathcal{O}_X$-module.

We write $\pi_X\colon T^*X \to X$ for the cotangent bundle to a smooth variety X. We identify $T^*(X \times X)$ with $T^*X \times T^*X$. However, we do this via the usual isomorphism $T^*(X \times X) \simeq T^*X \times T^*X$ composed with the antipode map on the right. This is dictated by requiring that the conormal bundle to the diagonal in $X \times X$ be identified with the diagonal in $T^*X \times T^*X$.

3. Convolution

Let G be a connected reductive group. Fix a Borel subgroup $B \subseteq G$. Then B acts on G via $b \cdot g = gb^{-1}$. The quotient under this action is the flag variety G/B. Clearly, B acts on G/B on the left, and we may form the equivariant derived category $D_m^b(B\backslash G/B)$.

Let $\tilde{q}\colon G \times G/B \to G/B$ be the projection on G composed with the quotient map $G \to G/B$. Let $p\colon G \times G/B \to G/B$ denote projection on G/B. Further, write $q\colon G \times G/B \to G \overset{B}{\times} G/B$ for the quotient map. Let $m\colon G \overset{B}{\times} G/B \to G/B$ denote the map induced by the action of G on G/B.

$$
\begin{array}{ccccc}
 & & G \times G/B & \xrightarrow[p]{q} G \overset{B}{\times} G/B & \xrightarrow{m} G/B \\
 & \tilde{q} \nearrow & & \searrow & \\
G/B & \leftarrow & & G/B &
\end{array}
$$

The *convolution* bifunctor $- \cdot -\colon D^b_m(B\backslash G/B) \times D^b_m(B\backslash G/B) \to D^b_m(B\backslash G/B)$ is defined by the formula

$$M \cdot N = m_!(M \widetilde{\boxtimes} N),$$

where $M \widetilde{\boxtimes} N$ denotes the descent of $\tilde{q}^* M \otimes p^* N[d_B]$ to $D^b_m(B\backslash G \overset{B}{\times} G/B)$. This is an associative operation and endows $D^b_m(B\backslash G/B)$ with a monoidal structure. Convolution adds weights and commutes with Verdier duality, since m is proper.

3.1. **Another description.** The group G acts on $G/B \times G/B$ diagonally. The map $G \times G/B \to G/B \times G/B$, $(g,x) \mapsto (\tilde{q}(g), g \cdot x)$ induces a G-equivariant isomorphism

$$G/B \times G/B \xrightarrow{\sim} G \overset{B}{\times} G/B.$$

Under this isomorphism $m\colon G \overset{B}{\times} G/B \to G/B$ corresponds to projection on the second factor $p_2\colon G/B \times G/B \to G/B$. Define $i\colon G/B \to G/B \times G/B$, $x \mapsto (\tilde{q}(1), x)$. Using equivariant descent (see [MV, Lemma 1.4]) we infer

(3.1.1) $\qquad i^*[-d_{G/B}]\colon D^b_m(G\backslash(G/B \times G/B)) \xrightarrow{\sim} D^b_m(B\backslash G/B)$

is a t-exact equivalence. If $M \in D^b_m(G\backslash(G/B \times G/B))$ is pure of weight n, then $i^* M[-d_{G/B}]$ is pure of weight $-d_{G/B}$.

Let $r = \mathrm{id}_{G/B} \times \Delta \times \mathrm{id}_{G/B}$, where $\Delta\colon G/B \to G/B \times G/B$ is the diagonal embedding. Define a monoidal structure $- \cdot -$ on $D^b_m(G\backslash(G/B \times G/B))$ by

(3.1.2) $\qquad M \cdot N = p_{13!} r^*(M \boxtimes N)[-d_{G/B}],$

where $p_{13}\colon G/B \times G/B \times G/B \to G/B \times G/B$ denotes projection on the first and third factor. A diagram chase (omitted) shows that the equivalence (3.1.1) is monoidal. We will constantly go back and forth between these two descriptions.

3.2. **Braid relations.** Fix a maximal torus $T \subseteq B$. Let $W = N_G(T)/T$ be the Weyl group. Write $\ell\colon W \to \mathbf{Z}_{\geq 0}$ for the length function. The B-orbits in G/B are indexed by W. Further, writing X_w for the orbit corresponding to $w \in W$, we have $X_w \simeq \mathbf{C}^{\ell(w)}$. For each $w \in W$, let $i_w\colon X_w \hookrightarrow G/B$ be the inclusion map. Set

$$\mathbf{T}_w = i_{w!}\underline{X_w} \quad \text{and} \quad \mathbf{C}_w = IC(X_w, \underline{X_w})[-\ell(w)].$$

Then $\mathbf{T}_e$ is the unit for convolution and will be denoted by $\mathbf{1}$. Both $\mathbf{T}_w[\ell(w)]$ and $\mathbf{C}_w[\ell(w)]$ are in $\mathrm{MHM}(B\backslash G/B)$.

Write Y_w for the image of $G \overset{B}{\times} X_w$ under the isomorphism (3.1.1). Then the Y_w, $w \in W$, are the G-orbits in $G/B \times G/B$. Furthermore, $\mathbf{T}_w = i^* j_{w!} \underline{Y_w}$ and $\mathbf{C}_w = i^* IC(Y_w, \underline{Y_w})[-\ell(w)]$, where $j_w \colon Y_w \hookrightarrow G/B \times G/B$ is the inclusion map.

3.3. **Proposition.** *If $\ell(ww') = \ell(w) + \ell(w')$, then $\mathbf{T}_w \cdot \mathbf{T}_{w'} = \mathbf{T}_{ww'}$*

Proof. If $\ell(ww') = \ell(w) + \ell(w')$, then $Y_{ww'} = Y_w \times_{G/B} Y_{w'}$, where the fibre product is over the projection maps $Y_w \to G/B$ and $Y_{w'} \to G/B$ on the first and second factor respectively. Now an application of proper base change and the description of convolution on $D^b_m(G\backslash(G/B \times G/B))$ yields the result. $\square$

3.4. **Lemma.** *$\mathbf{T}_w[\ell(w)] \cdot -$ is left t-exact, and $(\mathbf{DT}_w)[-\ell(w)] \cdot -$ is right t-exact.*

Proof. It suffices to show $\mathbf{T}_w[\ell(w)] \cdot -$ is left t-exact, since Verdier duality commutes with convolution. Consider the diagram

$$
\begin{array}{ccccc}
 & & BwB \times G/B & \xrightarrow[p_w]{q_w} & BwB \overset{B}{\times} G/B \xrightarrow{m_w} G/B \\
 & \overset{\tilde{q}_w}{\nearrow} & & \searrow & \\
X_w & \leftarrow & & & G/B
\end{array}
$$

where $\tilde{q}_w$ is the the evident quotient map on the first factor followed by projection, $p_w \colon BwB \times G/B \to G/B$ is projection on the second factor, q_w is the restriction of q, and m_w is the restriction of m. Then $\mathbf{T}_w \cdot - = m_{w!}(\underline{X_w} \widetilde{\boxtimes} -)$, where $\underline{X_w} \widetilde{\boxtimes} -$ is the descent of $\tilde{q}_w^* \underline{X_w} \otimes p_w^*(-)[d_B]$ to $D^b_m(B\backslash BwB \overset{B}{\times} G/B)$. Now $\underline{X_w}[\ell(w)] \widetilde{\boxtimes} -$ is t-exact. This implies the result, since m_w is affine. $\square$

3.5. **Proposition.** *Let $s \in W$ be a simple reflection and G/P_s the corresponding partial flag variety. Let $\pi_s \colon G/B \to G/P_s$ be the projection. Then $\mathbf{C}_s \cdot M = \pi_s^* \pi_{s*} M$, for all $M \in \mathscr{H}$.*

Proof. The closure $\overline{Y_s}$ of Y_s in $G/B \times G/B$ is smooth (it is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$). Hence, $IC(Y_s, \underline{Y_s}) = \tilde{\imath}_! \underline{\overline{Y_s}}[d_{\overline{Y_s}}]$, where $\tilde{\imath} \colon \overline{Y_s} \hookrightarrow G/B \times G/B$ is the inclusion. Further, π_s is a Zariski locally trivial $\mathbf{P}^1$-fibration. Using proper base change we deduce

$$(j_{s!*} \underline{Y_s}[d_{Y_s} - 1]) \cdot N = (\mathrm{id}_{G/B} \times \pi_s)^* (\mathrm{id}_{G/B} \times \pi_s)_* N$$

for all $N \in D^b_m(G\backslash(G/B \times G/B))$. Further, if $\tilde{M} \in D^b_m(G\backslash(G/B \times G/B))$ is such that $i^* \tilde{M}[-d_{G/B}] = M$, then

$$i^*(\mathrm{id}_{G/B} \times \pi_s)^* (\mathrm{id}_{G/B} \times \pi_s)_* \tilde{M}[-d_{G/B}] = \pi_s^* \pi_{s*} M. \qquad \square$$

3.6. **Corollary.** *Let $s \in W$ be a simple reflection. Then $\mathbf{C}_s \cdot \mathbf{C}_s = \mathbf{C}_s \oplus \mathbf{C}_s[-2](-1)$.*

Let

$$\mathbf{i}\colon D_m^b(B\backslash G/B) \xrightarrow{\sim} D_m^b(B\backslash G/B)$$

denote the auto-equivalence induced by the automorphism of $G/B \times G/B$ that switches the factors. Then $\mathbf{i}(M \cdot N) = \mathbf{i}N \cdot \mathbf{i}M$, for all $M, N \in D_m^b(B\backslash G/B)$. Further, if $s \in W$ is a simple reflection, then $\mathbf{i}\mathbf{T}_s = \mathbf{T}_s$. Consequently, $\mathbf{i}\mathbf{T}_w = \mathbf{T}_{w^{-1}}$ and $\mathbf{i}\mathbf{C}_w = \mathbf{C}_{w^{-1}}$ for all $w \in W$.

3.7. Proposition. *Let $s \in W$ be a simple reflection. Then*

    (i) $\mathbf{C}_s \cdot \mathbf{T}_s = \mathbf{C}_s[-2](-1) = \mathbf{T}_s \cdot \mathbf{C}_s$;

    (ii) $\mathbf{T}_s \cdot \mathbf{DT}_s = \mathbf{1} = \mathbf{DT}_s \cdot \mathbf{T}_s$.

Proof. Prop. 3.5 gives the first equality in (i), and applying the involution $\mathbf{i}$ gives the second equality. Convolve the distinguished triangle $\mathbf{DT}_s \to \mathbf{1}[1] \to \mathbf{C}_s[3](1) \rightsquigarrow$ with $\mathbf{T}_s$, and use (i) along with Lemma 3.4 to get a short exact sequence

$$0 \to \mathbf{T}_s \cdot \mathbf{DT}_s \to \mathbf{T}_s[1] \to \mathbf{C}_s[1] \to 0$$

in $\mathrm{MHM}(B\backslash G/B)$. This implies $\mathbf{T}_s \cdot \mathbf{DT}_s = \mathbf{1}$. That $\mathbf{DT}_s \cdot \mathbf{T}_s = \mathbf{1}$ follows from Verdier duality. $\square$

Combining Prop. 3.3 with Prop. 3.7(ii) yields:

3.8. Theorem. *Each $\mathbf{T}_w$, $w \in W$, is invertible under convolution.*

4. Action on coherent sheaves

Let X be a smooth variety and $\pi_X\colon T^*X \to X$ its cotangent bundle.

4.1. Filtered $\mathcal{D}$-modules. Let $F_i(\mathcal{D}_X)$ denote the sub-sheaf of $\mathcal{D}_X$ consisting of differential operators of degree at most i. This defines a filtration of $\mathcal{D}_X$. Write $\mathrm{gr}\mathcal{D}_X$ for the associated graded sheaf of rings. Then we have a canonical isomorphism

$$\mathrm{gr}\mathcal{D}_X \simeq \pi_{X*}\mathcal{O}_{T^*X}.$$

Identify $\mathrm{gr}\mathcal{D}_X$ with $\pi_{X*}\mathcal{O}_{T^*X}$ via this isomorphism. A *filtered $\mathcal{D}_X$-module* is a pair (M, F), where M is a $\mathcal{D}_X$-module and F is an exhaustive filtration of M by sub-sheaves such that $F_i(\mathcal{D}_X)F_jM \subseteq F_{i+j}M$. The filtration F is a *good filtration* if $\mathrm{gr}(M)$ is coherent as a $\mathrm{gr}\mathcal{D}_X$-module. The support of $\mathcal{O}_{T^*X} \otimes_{\pi_X^{-1}\mathrm{gr}\mathcal{D}_X} \mathrm{gr}(M)$ is the *characteristic variety* of M.

A mixed Hodge module $M \in \mathrm{MHM}(X)$ is a tuple $(M, F, \mathrm{rat}(M), W)$, where M is a regular holonomic $\mathcal{D}_X$-module, F is a good filtration on M (the *Hodge filtration*), $\mathrm{rat}(M)$ is a perverse sheaf on X with $\mathbf{Q}$-coefficients (the *rational structure*) such that $\mathscr{DR}(M) = \mathbf{C} \otimes_{\mathbf{Q}} \mathrm{rat}(M)$, where $\mathscr{DR}$ is the de Rham functor, and W is the *weight filtration* on $(M, F, \mathrm{rat}(M))$. This data is required to satisfy several compatibilities

which we only recall as needed. Morphisms in $\mathrm{MHM}(X)$ respect the filtrations F and W strictly. Given $(M, F, \mathrm{rat}(M), W) \in \mathrm{MHM}(X)$,

$$M(n) = (M, F_{\bullet - n}, \mathbf{Q}(n) \otimes_{\mathbf{Q}} \mathrm{rat}(M), W_{\bullet + 2n}),$$

where $\mathbf{Q}(n) = (2\pi\sqrt{-1})^n \mathbf{Q}$.

The weight filtration W and rational structure $\mathrm{rat}(M)$ are not particularly relevant for us in this section. Consequently, we omit them from our notation from here on and focus on the filtered $\mathcal{D}$-module structure underlying a mixed Hodge module.

4.2. **The functor $\tilde{\mathrm{gr}}$.** Let $(M, F) \in \mathrm{MHM}(X)$. Taking the associated graded with respect to F gives a coherent $\mathrm{gr}(\mathcal{D}_X)$-module $\mathrm{gr}(M)$. Hence, we obtain an exact functor from $\mathrm{MHM}(X)$ to graded coherent $\mathrm{gr}(\mathcal{D}_X)$-modules. We have $\mathbf{C}^*$ acting on T^*X via dilation of the fibres of π_X. As π_X is affine, π_{X*} gives an equivalence between $\mathbf{C}^*$-equivariant quasi-coherent $\mathcal{O}_{T^*X}$-modules and graded quasi-coherent $\pi_{X*}\mathcal{O}_{T^*X}$-modules. Thus, we obtain an exact functor

$$\tilde{\mathrm{gr}} \colon \mathrm{MHM}(X) \to \mathrm{Coh}^{\mathbf{C}^*}(\mathcal{O}_{T^*X}), \quad M \mapsto \mathcal{O}_{T^*X} \otimes_{\pi_X^{-1}\mathrm{gr}\mathcal{D}_X} \pi_X^{-1}\mathrm{gr}(M),$$

with $\mathbf{C}^*$ action on $\tilde{\mathrm{gr}}(M)$ defined by

$$z \cdot (f(x, \xi) \otimes m_i) = f(x, z^{-1}\xi) \otimes z^{-i}m_i,$$

where $z \in \mathbf{C}^*$, $f(x, \xi) \in \mathcal{O}_{T^*X}$, and m_i is in the i-th component of $\mathrm{gr}(M)$.

4.3. **Tate twist and $\tilde{\mathrm{gr}}$.** For $n \in \mathbf{Z}$ let $\mathbf{q}^n \in \mathrm{Coh}^{\mathbf{C}^*}(\mathrm{pt})$ be the one dimensional $\mathbf{C}^*$-module with the action of $z \in \mathbf{C}^*$ given by multiplication by z^n. Let $a \colon T^*X \to \mathrm{pt}$ be the obvious map. For $M \in D^{\mathbf{C}^*}(\mathcal{O}_{T^*X})$, set $M(n) = a^*\mathbf{q}^{-n} \otimes_{\mathcal{O}_{T^*X}} M$. Evidently, if $N \in D^b_m(X)$, then

$$\tilde{\mathrm{gr}}(N(n)) = \tilde{\mathrm{gr}}(N)(n).$$

4.4. **Correspondences.** Let $f \colon X \to Y$ be a morphism of smooth varieties. Associated to f we have the diagram

$$T^*X \xleftarrow{f_d} T^*Y \times_Y X \xrightarrow{f_\pi} T^*Y,$$

where f_π is the base change of f along $T^*Y \to Y$, and f_d is the map dual to the derivative. Let $T^*_X X \subseteq T^*X$ denote the zero section. Set

$$T^*_X Y = f_d^{-1}(T^*_X X).$$

If f is the inclusion of a closed subvariety, then $T^*_X Y$ is the conormal bundle to X in T^*Y. Let $\Lambda \subseteq T^*Y$ be a conic (i.e. $\mathbf{C}^*$-stable) subvariety. Then f is *non-characteristic* for Λ if

$$f_\pi^{-1}(\Lambda) \cap T^*_X Y \subseteq T^*_Y Y \times_Y X.$$

This is equivalent to $f_d|_{f_\pi^{-1}(\Lambda)}$ being finite. We say f is non-characteristic for $M \in \mathrm{MHM}(X)$ if f is non-characteristic for the characteristic variety of M.

Let $X \xleftarrow{f} Z \xrightarrow{g} Y$ be a diagram of smooth varieties such that the canonical map $Z \to X \times Y$ is a closed immersion. We call such a diagram a *correspondence* between X and Y. Associated to a correspondence we have a functor

$$\Phi_{X|Y} \colon D_m^b(X) \to D_m^b(Y), \quad M \mapsto g_* f^* M.$$

We also have a commutative diagram

$$\begin{array}{ccc}
& q_X & & T_Z^*(X \times Y) & & q_Y \\
f_\pi & T^*X \times_X Z & f_d & & g_d & T^*Y \times_Y Z & g_\pi \\
T^*X & & & T^*Z & & & T^*Y
\end{array}$$

with middle square cartesian. So we obtain a correspondence

$$T^*X \xleftarrow{q_X} T_Z^*(X \times Y) \xrightarrow{q_Y} T^*Y.$$

For $M \in D^{\mathbf{C}^*}(\mathcal{O}_{T^*X})$ set

$$\tilde{\Phi}_{X|Y}(M) = q_{Y*}(q_X^* M \otimes_{\mathcal{O}_{T_Z^*(X \times Y)}} \rho^* \omega_{Z/Y})[d_{Z/X}](-d_{Z/Y}),$$

where $\rho \colon T_Z^*(X \times Y) \to Z$ is the evident map.

4.5. **Theorem** ([La2, Théorème 3.1.1]). *Let $X \xleftarrow{f} Z \xrightarrow{g} Y$ be a correspondence with g projective. If f is non-characteristic for $M \in D_m^b(X)$, then*

$$\tilde{\mathrm{gr}} \circ \Phi_{X|Y}(M) = \tilde{\Phi}_{X|Y} \circ \tilde{\mathrm{gr}}(M).$$

4.6. *Remark.* Let $f \colon X \to Y$ be a morphism between smooth varieties, and write $\Gamma_f \subseteq X \times Y$ for the graph of f. Let $p_X \colon \Gamma_f \to X$ and $p_Y \colon \Gamma_f \to Y$ be the projection maps. Then we have the correspondence

$$X \xleftarrow{p_X} \Gamma_f \xrightarrow{p_Y} Y,$$

and $\Phi_{X|Y} = f_*$. If f is projective, then Thm. 4.5 implies

$$\tilde{\mathrm{gr}} f_* = f_{\pi*} f_d^! \tilde{\mathrm{gr}}[d_{X/Y}](-d_{X/Y}).$$

Similarly, for the correspondence

$$Y \xleftarrow{p_Y} \Gamma_f \xrightarrow{p_X} X,$$

one has $\Phi_{Y|X} = f^*$. If f is smooth, then Thm. 4.5 implies

$$\tilde{\mathrm{gr}} f^* = f_{d*} f_\pi^* \tilde{\mathrm{gr}}[d_{X/Y}].$$

Although we have obtained the above formulae as consequences of Thm. 4.5, the proof of Thm. 4.5 proceeds by first obtaining these formulae. Further, in [La1] and [La2] the formulae do not keep track of the $\mathbf{C}^*$-equivariant structure. Regardless,

the (equivariant) formula for non-characteristic pullback is immediate from the definitions. The (equivariant) formula for proper pushforward requires a bit more work which is done in [Ta, Lemma 2.3]. With these in hand the proof of Thm. 4.5 proceeds exactly as that of [La2, Théorème 3.1.1]. We also note that [La1] and [La2] are written purely in the context of filtered $\mathcal{D}$-modules. In this generality [La2, Théorème 3.1.1] does not quite hold - a crucial 'strictness' assumption that is required for the formula for $\tilde{\mathrm{gr}} f_*$ is missing. However, this is not a problem for us: if the filtered $\mathcal{D}$-module structure is one underlying a mixed Hodge module, then this strictness assumption holds [Sa88, Théorème 1].

4.7. **The Steinberg variety.** Let $\pi\colon \tilde{\mathscr{N}} \to G/B$ denote the cotangent bundle of G/B. Then $G \times \mathbf{C}^*$ acts on $\tilde{\mathscr{N}}$ (the map π is G-equivariant and $\mathbf{C}^*$ acts via dilations of the fibres of π). Under our conventions, $\tilde{\mathscr{N}} \times \tilde{\mathscr{N}}$ is the cotangent bundle of $G/B \times G/B$. Further, $G \times \mathbf{C}^*$ acts on $\tilde{\mathscr{N}} \times \tilde{\mathscr{N}}$ via the diagonal action. The *Steinberg variety* $Z \subseteq \tilde{\mathscr{N}} \times \tilde{\mathscr{N}}$ is defined by

$$Z = \bigcup_{w \in W} T^*_{Y_w}(G/B \times G/B).$$

It is a closed $G \times \mathbf{C}^*$ stable subvariety of $\tilde{\mathscr{N}} \times \tilde{\mathscr{N}}$. The projection $Z \to \tilde{\mathscr{N}}$ to either of the two factors is projective.

Denote by $\tilde{p}_{13}\colon \tilde{\mathscr{N}} \times \tilde{\mathscr{N}} \times \tilde{\mathscr{N}} \to \tilde{\mathscr{N}} \times \tilde{\mathscr{N}}$ the projection on the first and third factor, and let $p_2\colon \tilde{\mathscr{N}} \times \tilde{\mathscr{N}} \times \tilde{\mathscr{N}} \to \tilde{\mathscr{N}}$ be projection on the second factor. Define $\tilde{r}\colon \tilde{\mathscr{N}} \times \tilde{\mathscr{N}} \times \tilde{\mathscr{N}} \to \tilde{\mathscr{N}} \times \tilde{\mathscr{N}} \times \tilde{\mathscr{N}} \times \tilde{\mathscr{N}}$ by

$$\tilde{r} = \mathrm{id}_{\tilde{\mathscr{N}}} \times \Delta \times \mathrm{id}_{\tilde{\mathscr{N}}},$$

where $\Delta\colon \tilde{\mathscr{N}} \to \tilde{\mathscr{N}} \times \tilde{\mathscr{N}}$ is the diagonal embedding. Let $\tilde{\mathscr{H}} \subseteq D^{\mathbf{C}^*}(\mathcal{O}_{\tilde{\mathscr{N}} \times \tilde{\mathscr{N}}})$ be the full subcategory consisting of complexes whose cohomology sheaves are supported on Z. Define a bifunctor $- \cdot -\colon \tilde{\mathscr{H}} \times \tilde{\mathscr{H}} \to \tilde{\mathscr{H}}$ by the formula

$$M \cdot N = \tilde{p}_{13*}\tilde{r}^*(M \boxtimes N).$$

This endows $\tilde{\mathscr{H}}$ with a monoidal structure. The unit is $\Delta_* \mathcal{O}_\Delta$.

Define $\gamma\colon D^b(G\backslash(G/B \times G/B)) \to \tilde{\mathscr{H}}$ by

$$\gamma(M) = \tilde{\mathrm{gr}}\mathrm{For}(M) \otimes_{\mathcal{O}_{\tilde{\mathscr{N}} \times \tilde{\mathscr{N}}}} \Delta_* \pi^* \omega_{G/B}^{-1}(-d_{G/B}).$$

4.8. **Theorem.** γ *is monoidal.*

Proof. That γ preserves the unit object can be seen directly. Now apply Thm. 4.5 to the correspondence

$$G/B \times G/B \times G/B \times G/B \xleftarrow{r} G/B \times G/B \times G/B \xrightarrow{p_{13}} G/B \times G/B.$$

The G-equivariance of M and N implies that the characteristic variety of $M \boxtimes N$ is contained in Z. Further, r is non-characteristic for $M \boxtimes N$. Consequently,

$$\gamma(M \cdot N) = \gamma(M) \cdot \gamma(N).$$

To complete the proof we need to argue that the associativity constraints on both sides are compatible. The associativity constraint on either side is defined via the usual adjunction maps and base change (iso)morphisms. These are compatible with each other by [La2, §2.6]. □

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E-mail address: `rsvirk@gmail.com`