Derived Equivalences and Category $\mathcal{O}$

By
Rahbar Virk

A dissertation submitted in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
(Mathematics)

at the
UNIVERSITY OF WISCONSIN – MADISON
2011

Acknowledgments

I cannot overstate the enormous intellectual debt I owe my teacher Arun Ram. Needless to say, this work would not have been possible without his guidance, patience and support.

Contents

Introduction

This work is concerned with the study of a family of derived auto-equivalences of the category $\mathcal{O}$ associated to a finite dimensional, complex, simple Lie algebra $\mathfrak{g}$. For quick reference, the main results are: Thm. 5.4, Thm. 7.10 and Thm. 9.4.

Category $\mathcal{O}$ sits at the intersection of several areas of mathematics: the study of Harish-Chandra modules of a complex algebraic group [BG], the theory of D-modules [BB81], the theory of perverse sheaves [So90], categorification [KMS], construction of knot invariants [St] (these references are not intended to be comprehensive). In this document I study a family of auto-equivalences of the derived category of the principal block of $\mathcal{O}$ and explain how much of the structure of this block is encoded in the combinatorics of these equivalences. *A posteriori*, it is now clear to me that this family of auto-equivalences has been studied by several authors in the past, albeit in various disguised forms.

Let me describe the contents of this document, section by section.

Section 1: The purpose of this section is to setup some notation, establish sign conventions and precisely define the type of categories/2-categories that will appear in the sequel. This avoids all possible confusion as to 'how strict' structures are required to be.

Section 2: This section recalls the formalism of adjoint functors. Everything in this section, is contained, at least implicitly, in [MacL, Ch. 4 §7]).

Section 5: The objective of this section is Thm. 5.4 which describes a situation wherein an identity at the level of the Grothendieck group of an abelian category can be lifted to an auto-equivalence of the derived category. This result is an abstraction of [Ri, Thm. 2.1] (also see [Ro, §2.2.3], [ABG, Lemma 4.1.1] and [Vo, Thm. 7.3.16]).

Section 6: This section is a rapid introduction to the category $\mathcal{O}$ associated to a complex, finite dimensional, simple Lie algebra. Almost all the material in this section can be found in [BGG], [BGS], [So98], [Bou68], [Bou75] and [Dix]. There is also now a book [HumCatO] devoted to representations in category $\mathcal{O}$.

Section 7: Following [Ja, §2.10], translation functors and wall crossing functors are considered. Thm. 5.4 then gives mutually inverse equivalences

$$\Theta_s^*, \Theta_s^! \colon \mathrm{D^b}(\mathcal{O}_0) \to \mathrm{D^b}(\mathcal{O}_0),$$

where $\mathcal{O}_0$ is the principal block of category $\mathcal{O}$. In Thm. 7.10 these equivalences are exploited to give elementary and uniform proofs of several results in the literature such as [GJ, §5.2.1], [Sc, Lemma 5.17], [Ca, Thm. 3.18] and [Bott, Thm. 15].

Section 8: The purpose of this section is to establish the braid relations for the functors Θ_s^* and $\Theta_s^!$. The main result is Thm. 8.6. Although this is the most interesting and difficult property of the functors $\Theta_s^*, \Theta_s^!$ (considered in this work), I don't work very hard for it: the result is deduced from the existing literature. I outline three different ways to do this.

Section 9: In this section I explain how to deduce Soergel's character formulas for tilting modules [So98, Thm. 6.7] and the Ringel self duality of the principal block of category $\mathcal{O}$ (implicit in [So98]) from the preceding analysis. The main result is Thm. 9.4. The character formula for tilting modules and the Ringel self duality of the principal block of $\mathcal{O}$ are obtained as Cor. 9.5 and Cor. 9.6, respectively. Similar results have been obtained in the setting of D-modules [BeGi, Thm. 6.10] and perverse sheaves [BBM, §2.3]. In fact, the proof of Thm. 9.4 mimics the proof of [BBM, Prop. 2.3].

1. Recollections on categories and functors

1.1. **Notions concerning functors.** Let $F, G : \mathcal{A} \to \mathcal{B}$ be functors between categories $\mathcal{A}$ and $\mathcal{B}$. A morphism of functors $\phi \colon F \to G$ consists of a morphism $\phi_X \colon F(X) \to G(X)$ for each $X \in \mathcal{A}$, such that $\phi_Y \circ F(f) = G(f) \circ \phi_X$ for every morphism $f \colon X \to Y$. The terms 'functorial', 'natural' and 'canonical' will be used as synonyms for 'a morphism of functors'. The identity endomorphism of a functor F will be denoted $\mathbb{1}_F$.

A functor $F \colon \mathcal{A} \to \mathcal{B}$ is *full* if the map it induces on Hom sets is surjective; it is *faithful* if the induced map is injective. It is an *equivalence* if there exists a functor $G : \mathcal{B} \to \mathcal{A}$ such that FG and GF are canonically isomorphic to $\mathrm{id}_{\mathcal{B}}$ and $\mathrm{id}_{\mathcal{A}}$, respectively. In this situation the functors F and G are *mutually inverse equivalences*. An equivalence is necessarily full and faithful. Moreover:

Proposition 1.1. *Let $F \colon \mathcal{A} \to \mathcal{B}$ be a full and faithful functor. Then F is an equivalence if and only if every object $Y \in \mathcal{B}$ is isomorphic to $F(X)$ for some $X \in \mathcal{A}$.*

Proof. See [KaSc06, Prop. 1.3.13]. □

Let $\mathcal{A}$ be a category. Let $\mathcal{S}\mathrm{et}$ be the category of sets. Let $\mathrm{Funct}(\mathcal{A}, \mathcal{S}\mathrm{et})$ be the category of functors $\mathcal{A} \to \mathcal{S}\mathrm{et}$. A functor $F \in \mathrm{Funct}(\mathcal{A}, \mathcal{S}\mathrm{et})$ is *representable* if $F \simeq \mathrm{Hom}_{\mathcal{A}}(X, -)$ for some object $X \in \mathcal{A}$. In this situation, the object X is said to *represent* F.

Lemma 1.2 (Yoneda lemma)**.** *The functor $\mathcal{A} \to \mathrm{Funct}(\mathcal{A}, \mathcal{S}\mathrm{et})$, $X \mapsto \mathrm{Hom}_{\mathcal{A}}(X, -)$ defines an equivalence of $\mathcal{A}$ with the full subcategory of representable functors.*

Proof. See [KaSc06, Prop. 1.4.3]. □

1.2. **Additive categories.** A category $\mathcal{A}$ is *additive* if all Hom sets are equipped with an abelian group structure such that composition of morphisms is bilinear and if all finite products exist in $\mathcal{A}$. The empty product gives a terminal object in $\mathcal{A}$. For $X, Y \in \mathcal{A}$, the maps $X \xleftarrow{\mathrm{id}} X \xrightarrow{0} Y$ give a unique map $X \to X \times Y$. Similarly, there is a unique map $Y \to X \times Y$. Consequently, finite products coincide with the corresponding coproducts. In particular, the terminal object is also initial and is hence a zero object.

Let $\mathcal{B}$ be another additive category. An *additive functor* $\mathcal{A} \to \mathcal{B}$ is a functor F such that $F(f+g) = F(f) + F(g)$ for all morphisms $f, g \in \mathcal{A}$. We write $\mathscr{H}om(\mathcal{A}, \mathcal{B})$ for the category of additive functors from $\mathcal{A}$ to $\mathcal{B}$. Functors between additive categories will always be assumed to be additive.

1.3. **Abelian categories.** An additive category is *abelian* if it possesses all kernels, cokernels and if every monomorphism is the kernel of some morphism and every epimorphism is the cokernel of some morphism. See [KaSc06, Ch. 8] for details. Let $\mathcal{A}$ be an abelian category. A sequence of maps $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_n} X_{n+1}$, in $\mathcal{A}$, is an *exact sequence* if the image of f_i is equal to the kernel of f_{i+1} for each $0 \le i < n$. An exact sequence $0 \to X \to Y \to Z \to 0$ is also referred to as a *short exact sequence*.

Let $\mathcal{B}$ be another abelian category. A functor $F \colon \mathcal{A} \to \mathcal{B}$ is *left exact* if for each exact sequence $0 \to X \to Y$ in $\mathcal{A}$, the sequence $0 \to F(X) \to F(Y)$ is exact in $\mathcal{B}$. Similarly, F is *right exact* if for each exact sequence $X \to Y \to 0$ in $\mathcal{A}$, the sequence $F(X) \to F(Y) \to 0$ is exact in $\mathcal{B}$. The functor F is *exact* if it is both left and right exact. The category of exact functors from $\mathcal{A}$ to $\mathcal{B}$ will be denoted $\mathscr{H}om_{\mathfrak{Ab}}(\mathcal{A}, \mathcal{B})$.

The *Grothendieck group* $K_0(\mathcal{A})$ is the free abelian group on symbols $[X]$, $X \in \mathcal{A}$, modulo the relation

$$[X] = [X_1] + [X_2]$$

for each short exact sequence $0 \to X_1 \to X \to X_2 \to 0$. Consequently, if

$$X^\bullet = \cdots \to X^i \to \cdots$$

is a bounded complex in $\mathcal{A}$, then $\sum_i (-1)^i [X^i] = \sum_i (-1)^i [H^i(X^\bullet)]$ in $K_0(\mathcal{A})$. If $F \in \mathscr{H}om_{\mathfrak{Ab}}(\mathcal{A}, \mathcal{B})$, then the map $[X] \mapsto [F(X)]$ is a group homomorphism $K_0(\mathcal{A}) \to K_0(\mathcal{B})$.

Let $\{L_i\}$ be a set of objects in $\mathcal{A}$ such that the classes $[L_i]$ comprise a basis of $K_0(\mathcal{A})$. Then for $M \in \mathcal{A}$, we write $[M : L_i]$ for the coefficient of L_i when $[M]$ is expanded in terms of the basis $\{[L_i]\}$, i.e., $[M] = \sum_i [M : L_i][L_i]$.

A *simple object* or an *object of length one* is an object $L \in \mathcal{A}$ such that any monomorphism $A \to L$ is either 0 or an isomorphism. For $n \ge 2$, *objects of length n* are inductively defined to be those objects X that fit into an exact sequence $0 \to X' \to X \to L \to 0$, with X' of length $n-1$ and L simple. If every object in $\mathcal{A}$ has finite length, then the Jordan-Hölder theorem holds in $\mathcal{A}$, i.e., for an object $X \in \mathcal{A}$, the length of X is well defined and the simple objects that occur in a 'composition series' of X are unique up to isomorphism and permutation (see [KaSc06, Exer. 8.20]).

1.4. **Complexes.** Let $\mathcal{A}$ be an additive category. A *complex* $X^\bullet$ in $\mathcal{A}$ is the data of a $\mathbb{Z}$-graded object $X^\bullet = \bigoplus_{i \in \mathbb{Z}} X^i$, $X^i \in \mathcal{A}$ and a degree 1 endomorphism $d_X : X^\bullet \to X^\bullet$ such that $d_X^2 = 0$. This is usually visualized as a sequence of morphisms $\cdots \to X^i \xrightarrow{d_i} X^{i+1} \to \cdots$, such that $d_{i+1} \circ d_i = 0$ for each i. The object X^i is in degree i and the morphisms d_i are those induced by d_X. The endomorphism d_X is the *differential* of $X^\bullet$. If $\mathcal{A}$ is an abelian category, the cohomology $H^*(X^\bullet)$ of $X^\bullet$ is the sequence of objects (in $\mathcal{A}$): $H^i(X^\bullet) = \frac{\ker(d_i)}{\mathrm{im}(d_{i-1})}$.

A *chain map* is a graded morphism $f\colon X^\bullet \to Y^\bullet$ of degree 0 such that $d_Y f = f d_X$. Let $f, g : X^\bullet \to Y^\bullet$ be chain maps. Then f and g are *homotopic* if there exists a graded morphism $s : X^\bullet \to Y^\bullet$ of degree -1 such that $d_Y s + s d_X = f - g$. The map s is a *homotopy* between f and g. Further, we say that f and g are in the same *homotopy class*. In the setting of abelian categories, homotopic maps induce the same maps on cohomology (see [KaSc06, Lemma 12.2.2]).

Denote by $\mathrm{Comp}(\mathcal{A})$ the category of all complexes, by $\mathrm{Comp}^-(\mathcal{A})$ the category of bounded above complexes, by $\mathrm{Comp}^+(\mathcal{A})$ the category of bounded below complexes and by $\mathrm{Comp}^{\mathrm{b}}(\mathcal{A})$ the category of bounded complexes, in $\mathcal{A}$. As each object of $\mathcal{A}$ is a complex concentrated in degree 0 we obtain a full and faithful embedding $\mathcal{A} \hookrightarrow \mathrm{Comp}(\mathcal{A})$.

The *shift functor* $[1]\colon \mathrm{Comp}(\mathcal{A}) \to \mathrm{Comp}(\mathcal{A})$ is defined as follows: if $X^\bullet$ is a complex with differential d_i, then $(X^\bullet[1])^i = X^{i+1}$ with differential $d'_i = -d_{i+1}$. It is clear that $[1]$ is a self-equivalence of $\mathrm{Comp}(\mathcal{A})$. For $n \in \mathbb{Z}$, set $[n] = [1]^n$.

Let $X^\bullet, Y^\bullet$ be complexes in $\mathcal{A}$ with differentials d'_i and d''_i, respectively. Let $\phi\colon X^\bullet \to Y^\bullet$ be a chain map. The *cone* of ϕ is

$$\mathrm{cone}(\phi)^i = Y^i \oplus X^{i+1} \quad \text{with differential} \quad d_i = \begin{pmatrix} d''_i & \phi_{i+1} \\ 0 & -d'_{i+1} \end{pmatrix}. \tag{1.1}$$

Define

$$\iota\colon Y^\bullet \to \mathrm{cone}(\phi), \quad Y^i \xrightarrow{\binom{\mathrm{id}}{0}} Y^i \oplus X^{i+1},$$
$$\delta\colon \mathrm{cone}(\phi) \to X^\bullet[1], \quad Y^i \oplus X^{i+1} \xrightarrow{(0\ \mathrm{id})} X^{i+1}.$$

Both inc and δ are chain maps. A *standard triangle* is a sequence of morphisms of the form

$$X^\bullet \xrightarrow{\phi} Y^\bullet \xrightarrow{\iota} \mathrm{cone}(\phi) \xrightarrow{\delta} X^\bullet[1]. \tag{1.2}$$

1.5. **Monoidal categories.** A *monoidal category* is a category $\mathcal{C}$ equipped with a bifunctor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ such that the functors $(X, Y, Z) \mapsto (X \otimes Y) \otimes Z$ and $(X, Y, Z) \mapsto X \otimes (Y \otimes Z)$ are equal and if there exists an object $\mathbb{1} \in \mathcal{C}$ such that the functors $X \mapsto X \otimes \mathbb{1}$ and $X \mapsto \mathbb{1} \otimes X$ are equal to $\mathrm{id} : \mathcal{C} \to \mathcal{C}$. If $\mathcal{C}$ is a monoidal category then the objects of $\mathcal{C}$ form a monoid with $\mathbb{1}$ as a unit. It follows that $\mathbb{1}$ is unique.

Strictly speaking, examples of monoidal categories that arise 'in nature' are not monoidal in the above sense. The bifunctor $\otimes$ is usually not associative on the nose. Instead, one specifies the extra data of a collection of trifunctorial isomorphisms $c_{XYZ}\colon (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$ for all objects X, Y, Z of $\mathcal{C}$, referred to as the *associativity constraint*. The associativity constraint is required to satisfy the 'pentagon axiom': for any four objects W, X, Y, Z of $\mathcal{C}$, the pentagon formed by the five possible objects of $\mathcal{C}$ obtained by grammatically correct insertions of parentheses in the expression $W \otimes X \otimes Y \otimes Z$, and by the five isomorphisms between these objects arising from the associativity constraint, is a commutative diagram. Similarly, the equalities $\mathbb{1} \otimes X = X$ and $X \otimes \mathbb{1}$ are replaced by functorial isomorphisms required to satisfy to a commutative diagram involving three vertices. See [KaSc06, Ch. 4 §3] for more details.

Fortunately, every monoidal category, in the weak sense of the previous paragraph, is equivalent to a monoidal category in the strict sense of our definition [MacL, Ch. XI §3 Thm. 1]. This result will constantly be invoked to omit parentheses and the associativity and unit isomorphisms in formulas.

1.6. **2-categories.** A *2-category* $\mathfrak{C}$ is a class of objects along with the following data:

- for all objects $\mathcal{C}_1, \mathcal{C}_2 \in \mathfrak{C}$, a category $\mathscr{H}om_{\mathfrak{C}}(\mathcal{C}_1, \mathcal{C}_2)$, whose objects are *1-morphisms* of $\mathfrak{C}$ and whose morphisms are *2-morphisms* of $\mathfrak{C}$. Given 1-morphisms F and G, the set of 2-morphisms from F to G is denoted $\mathrm{Hom}_{\mathfrak{C}}(F, G)$;
- for all objects $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ a composition functor

$$\circ\colon \mathscr{H}om(\mathcal{C}_2, \mathcal{C}_3) \times \mathscr{H}om(\mathcal{C}_1, \mathcal{C}_2) \to \mathscr{H}om(\mathcal{C}_1, \mathcal{C}_3)$$

which is associative on the nose. Furthermore, for every $\mathcal{C} \in \mathfrak{C}$ we require that there exist a unit object $\mathrm{id}_{\mathcal{C}} \in \mathscr{H}om(\mathcal{C}, \mathcal{C})$. The meaning of this is clear from the following example: a 2-category with one object is the same as a monoidal category.

As for monoidal categories, one really should ask for associativity and the unit property only up to canonical isomorphism satisfying the relevant coherence conditions. However, just as for monoidal categories, once it has been verified that the 2-categories in question are well defined in this weak sense, they will be treated as if they were 2-categories in the strict sense.

A *2-functor* $F\colon \mathfrak{C} \to \mathfrak{C}'$ between 2-categories is the following collection of data:

- a map between objects $F\colon \mathfrak{C} \to \mathfrak{C}'$;
- functors $F\colon \mathscr{H}om_{\mathfrak{C}}(\mathcal{C}_1, \mathcal{C}_2) \to \mathscr{H}om_{\mathfrak{C}'}(F(\mathcal{C}_1), \mathcal{F}(\mathcal{C}_2))$ for all $\mathcal{C}_1, \mathcal{C}_2 \in \mathfrak{C}$;
- natural equivalences $c_{XY}\colon F(X \circ Y) \xrightarrow{\sim} F(X) \circ F(Y)$ for all 1-morphisms X, Y (whenever the composition $X \circ Y$ makes sense), such that

$$c_{X\mathbb{1}} = c_{\mathbb{1}X} = \mathrm{id}_X$$

and the following diagram commutes:

$$\begin{array}{ccccc} F((X \circ Y) \circ Z) & \xrightarrow{c_{(X\circ Y)Z}} & F(X \circ Y) \circ F(Z) & \xrightarrow{c_{XY} \circ \mathrm{id}} & (F(X) \circ F(Y)) \circ F(Z) \\ \| & & & & \| \\ F(X \circ (Y \circ Z)) & \xrightarrow{c_{X(Y\circ Z)}} & \mathcal{F}(X) \circ F(Y \circ Z) & \xrightarrow{\mathrm{id} \circ c_{YZ}} & F(X) \circ (F(Y) \circ F(Z)) \end{array}$$

2. Adjoint functors

2.1. **Adjunctions.** Let $f_*\colon \mathcal{A} \to \mathcal{B}$ and $f^*\colon \mathcal{B} \to \mathcal{A}$ be functors. An *adjunction* (f^*, f_*) between f^* and f_* is the data of two natural transformations $\varepsilon\colon f^* f_* \to \mathrm{id}_{\mathcal{A}}$ and $\eta : \mathrm{id}_{\mathcal{B}} \to f_* f^*$ such that the compositions

$$f_* \xrightarrow{\eta \mathbb{1}_{f_*}} f_* f^* f_* \xrightarrow{\mathbb{1}_{f_*} \varepsilon} f_* \qquad \text{and} \qquad f^* \xrightarrow{\mathbb{1}_{f^*} \eta} f^* f_* f^* \xrightarrow{\varepsilon \mathbb{1}_{f^*}} f^* \tag{2.1}$$

are equal to the identity on f_* and f^*, respectively. The morphisms η and ε are the *unit* and *counit* of the adjunction respectively.

An adjunction gives an isomorphism, functorial in $A \in \mathcal{A}$ and $B \in \mathcal{B}$:

$$\alpha_{A,B}\colon \mathrm{Hom}_{\mathcal{A}}(f^* B, A) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{B}}(B, f_* A), \qquad \phi \mapsto \mathbb{1}_{f_*} \phi \circ \eta_B.$$

The inverse is given by $\psi \mapsto \varepsilon_A \circ \mathbb{1}_{f^*}\psi$. Conversely, such a functorial isomorphism $\alpha_{A,B}$ provides an adjunction (f^*, f_*). Namely, set

$$\varepsilon_A = \alpha^{-1}_{A,f_*A}(\mathrm{id}_{f_*A}), \quad \eta_B = \alpha_{f^*B,B}(\mathrm{id}_{f^*B}).$$

If (f^*, f_*) is an adjunction, then the functor f^* is *left adjoint* to f_* and the functor f_* is *right adjoint* to f^*.

Lemma 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories. Suppose (f^*, f_*) is an adjunction between functors $f^*\colon \mathcal{A} \to \mathcal{B}$ and $f_*\colon \mathcal{B} \to \mathcal{A}$.*

(i) *If $X \in \mathcal{A}$ is such that $f^*X \neq 0$, then the unit map $\eta_X\colon X \to f_*f^*X$ is non-zero.*
(ii) *If $Y \in \mathcal{B}$ is such that $f_*Y \neq 0$, then the counit map $\varepsilon_Y\colon f^*f_*Y \to Y$ is non-zero.*

Proof. As the composition $f^*X \xrightarrow{f^*(\eta_X)} f^*f_*f^*X \xrightarrow{\varepsilon_{f^*X}} f^*X$ is the identity on f^*X (see (2.1)), we infer that if $f^*X \neq 0$, then $\eta_X \neq 0$. The proof of (ii) is similar. □

2.2. **Transpose maps.** Let (f^*, f_*) and (g^*, g_*) be adjunctions between functors $f^*, g^*\colon \mathcal{A} \to \mathcal{B}$ and $f_*, g_*\colon \mathcal{B} \to \mathcal{A}$. Let η and ε denote the unit and counit of the adjunction (f^*, f_*), and let η' and ε' denote the unit and counit of the adjunction (g^*, g_*). Let $\phi\colon f_* \to g_*$ be a natural transformation. The *transpose* $\phi^\vee : g^* \to f^*$ is the composition

$$g^* \xrightarrow{\mathbb{1}_{g^*}\eta} g^*f_*f^* \xrightarrow{\mathbb{1}_{g^*}\phi\mathbb{1}_{f^*}} g^*g_*f^* \xrightarrow{\varepsilon'\mathbb{1}_{f^*}} f^*. \tag{2.2}$$

The following is a reformulation of [MacL, Ch. 4 §7, Thm. 2].

Proposition 2.2. *Let (f^*, f_*) and (g^*, g_*) be adjunctions between $f^*, g^*\colon \mathcal{A} \to \mathcal{B}$ and $f_*, g_*\colon \mathcal{B} \to \mathcal{A}$. Let*

$$\alpha\colon \mathrm{Hom}_{\mathcal{A}}(f^*-,-) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{B}}(-,f_*-), \quad \alpha'\colon \mathrm{Hom}_{\mathcal{A}}(g^*-,-) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{B}}(-,g_*-)$$

be the canonical isomorphisms obtained from this data. Let $\phi\colon f_* \to g_*$ be a natural transformation. Then $\phi^\vee : g^* \to f^*$ is the unique natural transformation such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}(f^*-,-) & \xrightarrow{\circ\phi^\vee} & \mathrm{Hom}_{\mathcal{A}}(g^*-,-) \\ {\scriptstyle\alpha}\downarrow{\scriptstyle\sim} & & {\scriptstyle\sim}\downarrow{\scriptstyle\alpha'} \\ \mathrm{Hom}_{\mathcal{B}}(-,f_*-) & \xrightarrow[\phi\circ]{} & \mathrm{Hom}_{\mathcal{B}}(-,g_*-) \end{array}$$

Proof. By definition, $\alpha'^{-1}(\phi \circ \alpha(?)) = \varepsilon' \circ \mathbb{1}_{g^*}\phi \circ \mathbb{1}_{g^*f_*}? \circ \mathbb{1}_{g^*}\eta$. Since all morphisms involved are natural transformations,

$$\begin{aligned} \varepsilon' \circ \mathbb{1}_{g^*}\phi \circ \mathbb{1}_{g^*f_*}? \circ \mathbb{1}_{g^*}\eta &= \varepsilon' \circ \mathbb{1}_{g^*g_*}? \circ \mathbb{1}_{g^*}\phi\mathbb{1}_{f^*} \circ \mathbb{1}_{g^*}\eta \\ &= ? \circ \varepsilon'\mathbb{1}_{f^*} \circ \mathbb{1}_{g^*}\phi\mathbb{1}_{f^*} \circ \mathbb{1}_{g^*}\eta \\ &= ? \circ \phi^\vee. \end{aligned}$$

So $\alpha'^{-1}(\phi \circ \alpha(?)) = ? \circ \phi^\vee$ which gives the commutativity of the diagram. As α and α' are isomorphisms, the natural transformation

$$\circ\phi^\vee\colon \mathrm{Hom}_{\mathcal{A}}(f^*-,-) \to \mathrm{Hom}_{\mathcal{A}}(g^*-,-)$$

is suitably unique. This implies the uniqueness of $\phi^\vee$ by the Yoneda lemma (Lemma 1.2). □

Proposition 2.3. *Let (f^*, f_*) and (g^*, g_*) be adjunctions between $f^*, g^* \colon \mathcal{A} \to \mathcal{B}$ and $f_*, g_* : \mathcal{B} \to \mathcal{A}$. Let $\phi \colon f_* \to g_*$ be a natural transformation.*

(i) *Let η, ε denote the unit and counit of (f^*, f_*) and let η', ε' be the unit and counit of (g^*, g_*). Then the following diagrams commute:*

$$\begin{array}{ccc} f^*f_* & \xrightarrow{\varepsilon} & \mathrm{id} \\ {\scriptstyle \phi^\vee \mathbb{1}_{f_*}}\uparrow & & \uparrow{\scriptstyle \varepsilon'} \\ g^*f & \xrightarrow[\mathbb{1}_{g^*}\phi]{} & g^*g_* \end{array} \qquad \begin{array}{ccc} f_*f^* & \xrightarrow{\phi \mathbb{1}_{f^*}} & g_*f^* \\ {\scriptstyle \eta}\uparrow & & \uparrow{\scriptstyle \mathbb{1}_{g_*}\phi^\vee} \\ \mathrm{id} & \xrightarrow[\eta']{} & g_*g^* \end{array}$$

(ii) *Assume $\mathcal{A}$ and $\mathcal{B}$ are additive. Let $\psi \colon f_* \to g_*$ be a natural transformation, then $(\phi + \psi)^\vee = \phi^\vee + \psi^\vee$.*

(iii) *Let (h^*, h_*) be an adjunction between functors $h^* \colon \mathcal{A} \to \mathcal{B}$ and $h_* \colon \mathcal{B} \to \mathcal{A}$. Further, let $\psi \colon g_* \to h_*$ be a natural transformation. Then $(\psi \circ \phi)^\vee = \phi^\vee \circ \psi^\vee$.*

Proof. (i) follows from the commutativity of the diagram in Prop. 2.2. (ii) follows from our standing assumption that functors between additive categories are additive, i.e., the induced maps on Hom groups are homomorphisms. (iii) follows from the uniqueness part of Prop. 2.2. □

Proposition 2.4. *Let (f^*, f_*) be an adjunction between functors $f^* \colon \mathcal{A} \to \mathcal{B}$ and $f_* \colon \mathcal{B} \to \mathcal{A}$.*

(i) $\mathbb{1}_{f_*}^\vee = \mathbb{1}_{f^*}$.

(ii) *Assume $\mathcal{A}$ and $\mathcal{B}$ are additive. Then $0^\vee = 0$.*

(iii) *If $e \colon f_* \to f_*$ is idempotent, i.e., $e^2 = e$, then $e^\vee \colon f^* \to f^*$ is also idempotent.*

Proof. Each of the equalities follows from the uniqueness part of Prop. 2.2. Details are left to the reader out of sheer laziness. □

2.3. **Composing adjoints.** Let (f^*, f_*) and (g^*, g_*) be adjunctions between functors $g^* \colon \mathcal{A} \to \mathcal{B}$, $g_* \colon \mathcal{B} \to \mathcal{A}$, $f^* \colon \mathcal{B} \to \mathcal{C}$ and $f_* \colon \mathcal{C} \to \mathcal{B}$. Then we have the data of four morphisms (units and counits):

$$\eta \colon \mathrm{id}_\mathcal{B} \to f_*f^*, \quad \varepsilon \colon f^*f_* \to \mathrm{id}_\mathcal{C}, \quad \eta' \colon \mathrm{id}_\mathcal{A} \to g_*g^*, \quad \varepsilon \colon g^*g_* \to \mathrm{id}_\mathcal{B}.$$

Let $\overline{\eta}$ be the composition $\mathrm{id}_\mathcal{A} \xrightarrow{\eta'} g_*g^* \xrightarrow{\mathbb{1}_{g_*}\eta\mathbb{1}_{g^*}} g_*f^*f_*g^*$ and let $\overline{\varepsilon}$ be the composition $f^*g^*g_*f_* \xrightarrow{\mathbb{1}_{f^*}\varepsilon'\mathbb{1}_{f_*}} f^*f_* \xrightarrow{\varepsilon} \mathrm{id}_\mathcal{B}$. It is well known that f^*g^* is left adjoint to g_*f_*. However, we will need a more precise version of this:

Lemma 2.5. *The natural transformations $\overline{\eta}$ and $\overline{\varepsilon}$ define an adjunction (f^*g^*, g_*f_*). Further,*

(i) *the natural transformation η' is the transpose of ε, i.e., $\varepsilon^\vee = \eta'$.*

(ii) *the natural transformation ε is the transpose of η', i.e., $(\eta')^\vee = \varepsilon$.*

Proof. We have

$$\begin{aligned} \mathbb{1}_{g_*f_*}\overline{\varepsilon} \circ \overline{\eta}\mathbb{1}_{g_*f_*} &= \mathbb{1}_{g_*f_*}\varepsilon \circ \mathbb{1}_{g_*f_*f^*}\varepsilon'\mathbb{1}_{f_*} \circ \mathbb{1}_{g_*}\eta\mathbb{1}_{g^*g_*f_*} \circ \eta'\mathbb{1}_{g_*f_*} \\ &= \mathbb{1}_{g_*f_*}\varepsilon \circ \mathbb{1}_{g_*}\eta\mathbb{1}_{f_*} \circ \mathbb{1}_{g_*}\varepsilon'\mathbb{1}_{f_*} \circ \eta'\mathbb{1}_{g_*f_*} \\ &= \mathbb{1}_{g_*f_*}, \end{aligned}$$

where the first equality is the definition of $\overline{\varepsilon}$ and $\overline{\eta}$, the second equality holds due to η and ε' being natural transformations and the last equality follows from the definition of unit/counit (2.1). The proof that $\bar{\varepsilon}\mathbb{1}_{f^*g^*} \circ \mathbb{1}_{f^*g^*}\overline{\eta} = \mathbb{1}_{f^*g^*}$ is similar. Thus, $\overline{\eta}$ and $\bar{\varepsilon}$ define an adjunction (f^*g^*, g_*f_*).

Further,

$$\varepsilon^\vee = \varepsilon \mathbb{1}_{f^*f_*} \circ \overline{\eta} = \varepsilon \mathbb{1}_{f^*f_*} \circ \mathbb{1}_{f^*}\eta \mathbb{1}_{f_*} \circ \eta' = \eta',$$

where the first equality is the definition of transpose (2.2), the second equality is the definition of $\overline{\eta}$ and the last equality follows from the definition of the unit/counit (2.1). Similarly,

$$(\eta')^\vee = \varepsilon \circ \mathbb{1}_{f^*}\varepsilon' \mathbb{1}_{f_*} \circ \mathbb{1}_{f^*f_*}\eta' = \varepsilon. \qquad \square$$

2.4. **Right transposes.** Let $(f_!, f^!)$ and $(g_!, g^!)$ be adjunctions between functors $f_!, g_! \colon \mathcal{A} \to \mathcal{B}$ and $f^!, g^! \colon \mathcal{B} \to \mathcal{A}$. Write η and ε for the unit and counit of $(f_!, f^!)$, and write η' and ε' for the unit and counit of $(g_!, g^!)$. Suppose $\psi \colon g_! \to f_!$ is a natural transformation. Then the *right transpose* ${}^\vee\psi \colon f^! \to g^!$ is the composition

$$f^! \xrightarrow{\eta' \mathbb{1}_{f^!}} g^! g_! f^! \xrightarrow{\mathbb{1}_{g^!}\psi' \mathbb{1}_{f^!}} g^! f_! f^! \xrightarrow{\mathbb{1}_{g^!}\varepsilon} g^!. \tag{2.3}$$

The next result allows us to transport all the statements for transposes to right transposes.

Proposition 2.6. *Let $(f_!, f^!)$ and $(g_!, g^!)$ be adjunctions between functors $f_!, g_! \colon \mathcal{A} \to \mathcal{B}$ and $f^!, g^! \colon \mathcal{B} \to \mathcal{A}$. Let $\phi \colon f^! \to g^!$ be a natural transformation. Then ${}^\vee(\phi^\vee) = \phi$. Similarly, if $\psi \colon g_! \to f_!$ is a natural transformation, then $({}^\vee\psi)^\vee = \psi$*

Proof. Let η, ε be the unit and counit of $(f_!, f^!)$ and let η', ε' be the unit and counit of $(g_!, g^!)$. Then

$$\begin{aligned}
{}^\vee(\phi^\vee) &= \mathbb{1}_{g^!}\varepsilon \circ \mathbb{1}_{g^!}\varepsilon' \mathbb{1}_{f_!f^!} \circ \mathbb{1}_{g^!g_!}\phi \mathbb{1}_{f_!f^!} \circ \mathbb{1}_{g^!g_!}\eta \mathbb{1}_{f^!} \circ \eta' \mathbb{1}_{f^!} \\
&= \mathbb{1}_{g^!}\varepsilon \circ \mathbb{1}_{g^!}\varepsilon' \mathbb{1}_{f_!f^!} \circ \mathbb{1}_{g^!g_!}\phi \mathbb{1}_{f_!f^!} \circ \eta' \mathbb{1}_{f^!f_!f^!} \circ \eta \mathbb{1}_{f^!} \\
&= \mathbb{1}_{g^!}\varepsilon \circ \mathbb{1}_{g^!}\varepsilon' \mathbb{1}_{f_!f^!} \circ \eta' \mathbb{1}_{g^!f_!f^!} \circ \phi \mathbb{1}_{f_!f^!} \circ \eta \mathbb{1}_{f^!} \\
&= \mathbb{1}_{g^!}\varepsilon \circ \phi \mathbb{1}_{f_!f^!} \circ \eta \mathbb{1}_{f^!} \\
&= \phi \circ \mathbb{1}_{f^!}\varepsilon \circ \eta \mathbb{1}_{f^!} \\
&= \phi.
\end{aligned}$$

The first equality is by the definition of transpose (2.2) and right transpose (2.3), the second, third and fifth equalities are due to the fact that all morphisms involved are natural transformations. The fourth and last equalities follow from the definition of the unit/counit (2.1).

The proof that $({}^\vee\psi)^\vee = \psi$ is similar. $\square$

3. Triangulated categories

A triangulated category is an additive category along with an auto-equivalence, and a family of so-called *distinguished triangles* satisfying certain axioms. This subject deserves a whole book such as [Neeman]. I will not try to give an introduction to triangulated categories. However, I only assume that the reader is familiar with the axiomatics and basic properties of a triangulated category at the level of [KaSc06, Ch. 10 §1]. The purpose of this section is to fix some notation and to recall a few specific constructions.

The *shift functor* in a triangulated category will be denoted by [1]. For $n \in \mathbb{Z}$, set $[n] = [1]^n$. Let $\mathcal{T}$ be a triangulated category. A distinguished triangle

$$X \to Y \to Z \to X[1]$$

will often be written as

$$X \to Y \to Z \rightsquigarrow .$$

Further, Z will be referred to as the *cone* of the map $X \to Y$. Similarly, X will be referred to as the *cocone* of the map $Y \to Z$.

The *Grothendieck group* $K_0(\mathcal{T})$ is the free abelian group on symbols $[X]$, $X \in \mathcal{T}$, modulo the relation

$$[X] = [X_1] + [X_2]$$

for each distinguished triangle $X_1 \to X \to X_2 \rightsquigarrow$. In particular, $[X[1]] = -[X]$ (see [KaSc06, §10.1, TR3]).

Let $\mathcal{T}'$ be another triangulated category. A *triangulated* or an *exact* functor $\mathcal{T} \to \mathcal{T}'$ is the data of a functor F that preserves distinguished triangles and a canonical isomorphism $F \circ [1] \xrightarrow{\sim} [1] \circ F$. A morphism $F \to F'$ between triangulated functors is a natural transformation θ such that the following diagram commutes

$$\begin{array}{ccc} F \circ [1] & \xrightarrow{\theta \circ [1]} & F' \circ [1] \\ \sim\downarrow & & \downarrow\sim \\ [1] \circ F & \xrightarrow{[1] \circ \theta} & [1] \circ F' \end{array}$$

All natural transformations between triangulated functors will tacitly be assumed to be morphisms of triangulated functors.

Let $\mathcal{A}$ be an abelian category. A functor $H \colon \mathcal{T} \to \mathcal{A}$ is *cohomological* if, for every distinguished triangle $X \to Y \to Z \rightsquigarrow$ in $\mathcal{T}$, the sequence $H(X) \to H(Y) \to H(Z)$ is exact in $\mathcal{A}$.

Proposition 3.1. *Let $\mathcal{T}$ be a triangulated category and let $X \in \mathcal{T}$. The functors* $\mathrm{Hom}_{\mathcal{T}}(X, -)$ *and* $\mathrm{Hom}_{\mathcal{T}}(-, X)$ *are cohomological.*

Proof. See [KaSc06, Prop. 10.1.13]. □

3.1. **Filtrations.** Let $\mathcal{T}$ be a triangulated category. Let $\mathcal{A}, \mathcal{B} \subset \mathcal{T}$ be subcategories of $\mathcal{T}$. For $X \in \mathcal{T}$, write $[X] \in \mathcal{A}$ if there exists an object in $\mathcal{A}$ that is isomorphic to X. Set

$$\mathcal{A} * \mathcal{B} = \{Y \in \mathcal{T} \mid \text{there exists } X \to Y \to Z \rightsquigarrow, \text{ with } [X] \in \mathcal{A} \text{ and } [Z] \in \mathcal{B}\}.$$

Lemma 3.2 ([BBD, Lemme 1.3.10])**.** *The operation $*$ is associative. That is, if $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are subcategories of $\mathcal{T}$, then $(\mathcal{A} * \mathcal{B}) * \mathcal{C} = \mathcal{A} * (\mathcal{B} * \mathcal{C})$.*

Proof. Suppose $[X] \in (\mathcal{A} * \mathcal{B}) * \mathcal{C}$. Then there is some $X' \in \mathcal{T}$ and distinguished triangles $A \to X' \to B \rightsquigarrow$ and $X' \to X \to C \rightsquigarrow$, with $[A] \in \mathcal{A}$, $[B] \in \mathcal{B}$ and $[C] \in \mathcal{C}$. Apply the octahedron axiom (see [KaSc06, Def. 10.1.6 TR5]) to the composition $A \to X' \to X$ to obtain distinguished triangles $A \to X \to BC \rightsquigarrow$ and $B \to X'' \to C \rightsquigarrow$, with $X'' \in \mathcal{T}$. Thus, $[X] \in \mathcal{A} * (\mathcal{B} * \mathcal{C})$. The reverse inclusion is proved similarly. □

Let $\mathcal{A} \subseteq \mathcal{T}$ be a subcategory. Inductively define $\mathcal{A}^{*i}$, $i \in \mathbb{Z}_{\geq 0}$, by $\mathcal{A}^{*0} = 0$ and $\mathcal{A}^{*i+1} = \mathcal{A} * \mathcal{A}^{*i}$. As $*$ is associative, $\mathcal{A}^{*i+1} = \mathcal{A} * \mathcal{A}^{*i} = \mathcal{A}^{*i} * \mathcal{A}$. Further, $\mathcal{A}^{*i} \subseteq \mathcal{A}^{*i+1}$. Set $\mathcal{A}^{*\infty} = \bigcup_{i \in \mathbb{Z}_{\geq 0}} \mathcal{A}^{*i}$.

An object $X \in \mathcal{T}$ is *filtered* by objects $Y_1, \ldots, Y_n$ if there exists a sequence of objects $0 = X_0, X_1, \ldots, X_n = X$ and distinguished triangles $X_{i-1} \to X_i \to Y_i \rightsquigarrow$.

Lemma 3.3. *Let $\mathcal{A} \subset \mathcal{T}$ be a subcategory. Then $X \in \mathcal{T}$ is in $\mathcal{A}^{*n}$ if and only if X is filtered by some $Y_1, \ldots, Y_n \in \mathcal{A}$.*

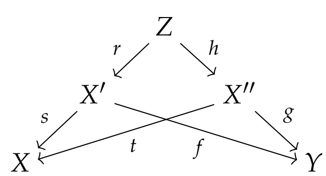
Proof. Proceeding by induction on n this is clear. $\square$

Remark 3.4. Filtrations in triangulated categories are most commonly used in the following situation: let H be a cohomological functor. Let X be filtered by $Y_1, \ldots, Y_n$. By definition, there is a sequence of objects $0 = X_0, \ldots, X_n = X$ and distinguished triangles $X_{i-1} \to X_i \to Y_i \rightsquigarrow$. Assume that $H(Y_i[m]) = 0$ for all $m \in \mathbb{Z}$ and $1 \leq i \leq n$. Then, proceeding by induction on n, it follows that $H(X[m]) = 0$ for all $m \in \mathbb{Z}$.

3.2. **Localization.** Let $\mathcal{T}$ be a triangulated category. Let $\mathcal{N} \subset \mathcal{T}$ be a *localizing* subcategory, i.e., $\mathcal{N}$ satisfies the following properties:

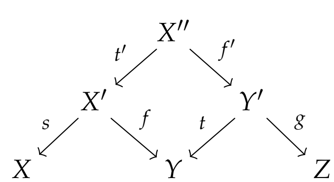
- $0 \in \mathcal{N}$;
- $N \in \mathcal{N}$ if and only if $N[1] \in \mathcal{N}$;
- if $N \to M \to N' \rightsquigarrow$ is a distinguished triangle in $\mathcal{T}$ with $N, N' \in \mathcal{N}$, then $M \in \mathcal{N}$.

An $\mathcal{N}$*-quasi-isomorphism*, or simply *quasi-isomorphism* if the $\mathcal{N}$ is clear, is a morphism $s\colon X \to Y$ in $\mathcal{T}$ such that there is a distinguished triangle $X \xrightarrow{s} Y \to Z \rightsquigarrow$ with $Z \in \mathcal{N}$. Let $\mathcal{N}-$qis denote the collection of $\mathcal{N}$-quasi-isomorphisms. A *roof* (s, f) is a diagram of the form $X \xleftarrow{s} X' \xrightarrow{f} Y$, with $s \in \mathcal{N}-$qis. Define an equivalence relation on roofs by declaring $X \xleftarrow{s} X' \xrightarrow{f} Y$ and $X \xleftarrow{t} X'' \xleftarrow{g} Y$ to be equivalent if there exists a third roof $X' \xleftarrow{r} Z \xrightarrow{h} X''$ such that the following diagram commutes



This equivalence relation is reflexive, symmetric and transitive (see [GeMa, Ch. 3 §2, Lemma 8 (a)] or [KaSc06, Lemma 7.1.12])

Given roofs $X \xleftarrow{s} X' \xrightarrow{f} Y$ and $Y \xleftarrow{t} Y' \xrightarrow{g} Z$, there is a roof $X' \xleftarrow{t'} X'' \xrightarrow{f'} Y'$ such that the following diagram commutes



The roof $X \xleftarrow{st'} X'' \xrightarrow{gf'} Z$ is defined to be the composition of $X \xleftarrow{s} X' \xrightarrow{f} Y$ and $Y \xleftarrow{t} Y' \xrightarrow{g} Z$. This operation is well defined and associative on equivalence classes of roofs. For details see [GeMa, Ch. 3 §2 Lemma 8 (b)] or [KaSc06, Lemma 7.1.13].

The *localization* of $\mathcal{T}$ with respect to $\mathcal{N}$, denoted $\mathcal{T}/\mathcal{N}$, is the following category:

- $\text{Objects}(\mathcal{T}/\mathcal{N}) = \text{Objects}(\mathcal{T})$;
- $\text{Hom}_{\mathcal{T}/\mathcal{N}}(X,Y) =$ equivalence classes of roofs $X \xleftarrow{s} X' \xrightarrow{f} Y$,

with composition of roofs defined as above.

The localization functor $\text{quot}\colon \mathcal{T} \to \mathcal{T}/\mathcal{N}$ is defined to be the identity on objects and by sending $f\colon X \to Y$ in $\mathcal{T}$ to the roof $X \xleftarrow{\text{id}} X \xrightarrow{f} Y$. We abuse notation and write $[1]\colon \mathcal{T}/\mathcal{N} \to \mathcal{T}/\mathcal{N}$ for the image of $[1]\colon \mathcal{T} \to \mathcal{T}$ under quot.

Proposition 3.5. *Define distinguished triangles in $\mathcal{T}/\mathcal{N}$ as sequences equivalent to the image (under* quot*) of a distinguished triangle in $\mathcal{T}$.*

(i) *$\mathcal{T}/\mathcal{N}$ is a triangulated category and* $\text{quot} : \mathcal{T} \to \mathcal{T}/\mathcal{N}$ *is a triangulated functor.*
(ii) *If $N \in \mathcal{N}$, then* $\text{quot}(N) = 0$.
(iii) *Let $\mathcal{T}'$ be a triangulated category and let $\mathcal{F} : \mathcal{T} \to \mathcal{T}'$ be a triangulated functor such that $F(N) = 0$ for each $N \in \mathcal{N}$. Then F factors uniquely through* quot.

Proof. See [KaSc06, Thm. 10.2.3]. □

3.3. **The homotopy category.** Let $\mathcal{A}$ be an additive category. The *homotopy category* of $\mathcal{A}$, denoted $\text{Ho}(\mathcal{A})$, is defined as follows:

- $\text{Objects}(\text{Ho}(\mathcal{A})) = \text{Objects}(\text{Comp}(\mathcal{A}))$;
- $\text{Hom}_{\text{Ho}(\mathcal{A})}(X^\bullet, Y^\bullet) =$ homotopy classes of maps in $\text{Hom}_{\text{Comp}(\mathcal{A})}(X^\bullet, Y^\bullet)$.

Replacing $\text{Comp}(\mathcal{A})$ by $\text{Comp}^+(\mathcal{A})$, $\text{Comp}^-(\mathcal{A})$ or $\text{Comp}^{\text{b}}(\mathcal{A})$ in the definition above we obtain the variants $\text{Ho}^+(\mathcal{A})$, $\text{Ho}^-(\mathcal{A})$ and $\text{Ho}^{\text{b}}(\mathcal{A})$, respectively.

Proposition 3.6. *Let $[1]\colon \text{Ho}(\mathcal{A}) \to \text{Ho}(\mathcal{A})$ be the shift functor on complexes. Define distinguished triangles in $\text{Ho}(\mathcal{A})$ to be triangles isomorphic to* (1.2)*. This endows $\text{Ho}(\mathcal{A})$ with the structure of a triangulated category.*

Proof. See [KaSc06, Thm. 11.2.6]. □

Proposition 3.7. *Let $\mathcal{A}$ be an abelian category For $n \in \mathbb{Z}$, let $H^n\colon \text{Ho}(\mathcal{A}) \to \mathcal{A}$ be the functor that associates to a complex its n^{th} cohomology. Then H^n is cohomological.*

Proof. See [KaSc06, Cor. 12.2.5]. □

3.4. **The derived category.** Let $\mathcal{A}$ be an abelian category. Let $\mathcal{N} \subset \text{Ho}(\mathcal{A})$ be the subcategory consisting of complexes $X^\bullet$ such that $H^i(X^\bullet) = 0$ for all $i \in \mathbb{Z}$. Then $\mathcal{N}$ is a localizing subcategory (for details see [KaSc06, Ch. 13 §1]). So we are in the setting of Prop. 3.5. The *derived category* of $\mathcal{A}$, denoted $\text{D}(\mathcal{A})$, is the triangulated category $\text{Ho}(\mathcal{A})/\mathcal{N}$. Replacing $\text{Ho}(\mathcal{A})$ by $\text{Ho}^+(\mathcal{A})$, $\text{Ho}^-(\mathcal{A})$ or $\text{Ho}^{\text{b}}(\mathcal{A})$ in this definition, we obtain the variants $\text{D}^+(\mathcal{A})$, $\text{D}^-(\mathcal{A})$ and $\text{D}^{\text{b}}(\mathcal{A})$, respectively. The category $\text{D}^{\text{b}}(\mathcal{A})$ (resp. $\text{D}^+(\mathcal{A})$, resp. $\text{D}^-(\mathcal{A})$) is equivalent to the full subcategory of $\text{D}(\mathcal{A})$ consisting of complexes $X^\bullet$ such that $H^n(X^\bullet) = 0$ for $|n| >> 0$ (resp. $n << 0$, resp. $n >> 0$), see [KaSc06, Prop. 13.1.12] for details.

Proposition 3.8. *Let* $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *be an exact sequence in* $\mathrm{Comp}(\mathcal{A})$*. Then there exists a distinguished triangle* $X \xrightarrow{f} Y \xrightarrow{g} Z \rightsquigarrow$ *in* $\mathrm{D}(\mathcal{A})$*.*

Proof. See [KaSc06, Prop. 13.1.13] □

Proposition 3.9. *For* $n \in \mathbb{Z}$*, let* $H^n\colon \mathrm{D}(\mathcal{A}) \to \mathcal{A}$ *be the functor that sends a complex to its* n^{th} *cohomology. Then* H^n *is cohomological.*

Proof. See [KaSc06, Prop. 13.1.5]. □

Let $X, Y \in \mathrm{D}(A)$. Set $\mathrm{Ext}^k_{\mathcal{A}}(X, Y) = \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(X, Y[k])$. An object $X \in \mathcal{A}$ is also an object of $\mathrm{D}(\mathcal{A})$, since X is a complex concentrated in degree 0.

Proposition 3.10. *Let* $X, Y \in \mathcal{A}$*. Then*

(i) $\mathrm{Ext}^k_{\mathcal{A}}(X, Y) = 0$ *for* $k < 0$*;*
(ii) $\mathrm{Ext}^0_{\mathcal{A}}(X, Y) \simeq \mathrm{Hom}_{\mathcal{A}}(X, Y)$*. That is, the natural functor* $\mathcal{A} \to \mathrm{D}(\mathcal{A})$ *is full and faithful.*

Proof. See [KaSc06, Prop. 13.1.10]. □

The embedding $\mathcal{A} \to \mathrm{D}^{\mathrm{b}}(\mathcal{A})$ induces a map $K_0(\mathcal{A}) \to K_0(\mathrm{D}^{\mathrm{b}}(\mathcal{A}))$. This map is an isomorphism, the inverse is given by $[X^\bullet] \mapsto \sum_{i \in \mathbb{Z}} (-1)^i [H^i(X^\bullet)]$. The groups $K_0(\mathcal{A})$ and $K_0(\mathrm{D}^{\mathrm{b}}(\mathcal{A}))$ are identified via this isomorphism.

3.5. **Yoneda** Ext. Let $X, Y \in \mathcal{A}$. Let $Z = 0 \to Y \to Z_1 \to \cdots \to Z_n \to X \to 0$ be an exact sequence in $\mathcal{A}$. Define $\theta(Z) \in \mathrm{Ext}^n_{\mathcal{A}}(X, Y)$ by the roof

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & X & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow \\
0 & \longrightarrow & Y & \longrightarrow & Z_1 & \longrightarrow & \cdots & \longrightarrow & Z_n & \longrightarrow & 0 \\
\downarrow & & \| & & \downarrow & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & Y & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

(The top vertical arrow is a quasi-isomorphism).

Proposition 3.11. *Each element of* $\mathrm{Ext}^n_{\mathcal{A}}(X, Y)$ *is of the form* $\theta(Z)$ *for some exact sequence* $Z = 0 \to Y \to Z_1 \to \cdots \to Z_n \to X \to 0$ *in* $\mathcal{A}$*. Further:*

(i) $\theta(Z) = 0$ *if and only if there exists a commutative diagram with exact rows*

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & Z_1 & \longrightarrow & Z_2' & \longrightarrow & \cdots & \longrightarrow & Z_n & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & Z_1 & \longrightarrow & Z_2 & \longrightarrow & \cdots & \longrightarrow & Z_n & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

(ii) *If* $Z' = 0 \to Y \to Z_1' \to \cdots \to Z_n' \to X \to 0$ *is another exact sequence in* $\mathcal{A}$*, then* $\theta(Z) = \theta(Z')$ *if and only if there exists a commutative diagram*

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & Y & \longrightarrow & Z_1 & \longrightarrow & \cdots & \longrightarrow & Z_n & \longrightarrow & X & \longrightarrow & 0 \\
 & & \| & & \downarrow & & & & \downarrow & & \| & & \\
0 & \longrightarrow & Y & \longrightarrow & Z_1' & \longrightarrow & \cdots & \longrightarrow & Z_n' & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

Proof. For the first statement and (i), see [KaSc06, Exer. 13.16] or [GeMa, Ch. III §5, Thm. 5 (c)]. (ii) is a restatement of the equivalence relation on roofs (see §3.2). ☐

Corollary 3.12. *Let $Z = 0 \to Y \to Z_1 \to X \to 0$ be a short exact sequence. Then $\theta(Z) \in \mathrm{Ext}^1_{\mathcal{A}}(X, Y)$ is zero if and only if Z is split exact.*

Let $Z' = 0 \to Y' \to Z_1 \to \cdots \to Z_m \to Y \to 0$ and $Z = 0 \to Y \to Z_{m+1} \to \cdots \to Z_{m+n} \to X \to 0$ be exact sequences in $\mathcal{A}$. Let $Z' \cup Z$ denote the exact sequence $0 \to Z_1 \to \cdots \to Z_m \to Z_{m+1} \to \cdots \to Z_{m+n} \to X \to 0$.

Proposition 3.13. $\theta(Z' \cup Z) = \theta(Z') \circ \theta(Z)$.

Proof. See [GeMa, Ch. 3 §5, Thm. 5 (c)]. ☐

3.6. **Projectives and injectives.** Let $\mathcal{A}$ be an abelian category. An object $P \in \mathcal{A}$ is *projective* if $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is exact. The category $\mathcal{A}$ has *enough projectives* if for any $A \in \mathcal{A}$ there exists an epimorphism $P \twoheadrightarrow A$ with P projective. Let $P_L \twoheadrightarrow L$ be an epimorphism with P_L projective and $L \in \mathcal{A}$ simple. Then P_L is a *projective cover* of L if P_L is indecomposable (i.e., P_L cannot be written as a non-trivial direct sum). A projective cover is unique up to isomorphism.

An object $I \in \mathcal{A}$ is *injective* if $\mathrm{Hom}_{\mathcal{A}}(-, I)$ is exact. The category $\mathcal{A}$ has *enough injectives* if for any $A \in \mathcal{A}$ there exists a monomorphism $A \hookrightarrow I$ with I injective. Let $L \hookrightarrow I_L$ be a monomorphism with I_L injective and $L \in \mathcal{A}$ simple. Then I_L is an *injective hull* of L if I_L is indecomposable. An injective hull is unique up to isomorphism.

Proposition 3.14. *Let $\mathcal{A}$ be an abelian category. Let $X \in \mathcal{A}$. The following are equivalent:*

(i) *X is projective.*
(ii) $\mathrm{Ext}^1_{\mathcal{A}}(X, Y) = 0$ *for all $Y \in \mathcal{A}$.*
(iii) $\mathrm{Ext}^n_{\mathcal{A}}(X, Y) = 0$ *for all $Y \in \mathcal{A}$ and all $n \neq 0$.*

Similarly, the following are equivalent:

(i) *X is injective.*
(ii) $\mathrm{Ext}^1_{\mathcal{A}}(Y, X) = 0$ *for all $Y \in \mathcal{A}$.*
(iii) $\mathrm{Ext}^n_{\mathcal{A}}(Y, X) = 0$ *for all $Y \in \mathcal{A}$ and all $n \neq 0$.*

Proof. See [GeMa, Ch. III §5, Lemma 10]. ☐

Remark 3.15. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $f_* \colon \mathcal{A} \to \mathcal{B}$ be right adjoint to $f^* \colon \mathcal{B} \to \mathcal{A}$. Assume that f_* is exact. Let $P \in \mathcal{B}$ be projective. Then f^*P is projective in $\mathcal{A}$, since $\mathrm{Hom}_{\mathcal{A}}(f^*P, -) \simeq \mathrm{Hom}_{\mathcal{B}}(P, f_*-)$. A similar statement holds for injectives.

Proposition 3.16. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{N} \subset \mathrm{Ho}(\mathcal{A})$ be the subcategory consisting of complexes $X^\bullet$ such that $H^i(X^\bullet) = 0$ for all $i \in \mathbb{Z}$. Let $\mathcal{I}$ be a full subcategory of $\mathcal{A}$ such that for any $X \in \mathcal{A}$, there exists $I \in \mathcal{I}$ and a monomorphism $X \hookrightarrow I$. Then*

(i) *for any $X \in \mathrm{Ho}^+(\mathcal{A})$, there exists $I \in \mathrm{Ho}^+(\mathcal{I})$ and a quasi-isomorphism $s \colon X \to I$;*
(ii) *let $\mathcal{N}' = \mathcal{N} \cap \mathrm{Ho}^+(\mathcal{I})$. The obvious functor $\mathrm{Ho}^+(\mathcal{I})/\mathcal{N}' \to \mathrm{D}^+(\mathcal{A})$ is a triangulated equivalence of categories.*

Proof. (i) is [KaSc06, Lemma 13.2.1], (ii) is [KaSc06, Prop. 13.2.2 (i)]. ☐

Lemma 3.17. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{I} \subseteq \mathcal{A}$ be the full subcategory consisting of injective objects. Let $I^\bullet \in \mathrm{Comp}^+(\mathcal{I})$. Let $X^\bullet \in \mathrm{Comp}(\mathcal{A})$ be such that the cohomology of $X^\bullet$ is zero in every degree. Let $f\colon X^\bullet \to I^\bullet$ be a chain map. Then f is homotopic to zero.*

Proof. See [KaSc06, Lemma 13.2.4]. □

Combining Prop. 3.16 and Lemma 3.17 we get:

Proposition 3.18. *Let $\mathcal{A}$ be an abelian category and let $\mathcal{I}$ be the full subcategory of $\mathcal{A}$ consisting of injective objects. If $\mathcal{A}$ has enough injectives, then $\mathrm{Ho}^+(\mathcal{I})$ is equivalent to $D^+(\mathcal{A})$ as a triangulated category.*

Proof. See [KaSc06, Prop. 13.2.3]. □

Assume we are in the situation of Prop. 3.16 (i), i.e., we are given a quasi-isomorphism $s\colon X \to I$ with $X \in \mathrm{Ho}^+(\mathcal{A})$ and $I \in \mathrm{Ho}^+(\mathcal{I})$, then I is a *resolution* of X by objects in $\mathcal{I}$.

3.7. **Derived functors.** Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and let $f_*\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor. A full additive subcategory $\mathcal{I}$ of $\mathcal{A}$ is *f_*-injective* if:

(i) for every object $X \in \mathcal{A}$ there is a monomorphism $X \hookrightarrow I$ with $I \in \mathcal{I}$;
(ii) if $0 \to X \to Y \to Z \to 0$ is an exact sequence in $\mathcal{A}$, and if X, Y are in $\mathcal{I}$, then Z is also in $\mathcal{I}$;
(iii) if $0 \to X \to Y \to Z \to 0$ is an exact sequence in $\mathcal{A}$ with $X, Y, Z \in \mathcal{I}$, then $0 \to f_*X \to f_*Y \to f_*Z \to 0$ is exact in $\mathcal{B}$.

If $\mathcal{A}$ has enough injectives, then the full subcategory of injective objects in $\mathcal{A}$ is f_*-injective for *any* left exact functor f_* (see [KaSc06, Remark 13.3.6 (iii)]).

Let $\mathcal{N} \subset \mathrm{Ho}(\mathcal{A})$ be the subcategory consisting of complexes whose cohomology vanishes in every degree. Suppose an f_*-injective subcategory $\mathcal{I} \subseteq \mathcal{A}$ exists. Set $\mathcal{N}' = \mathcal{N} \cap \mathrm{Ho}^+(\mathcal{I})$. Since f_* preserves exact sequences consisting of objects in $\mathcal{I}$, it follows that f_* transforms objects of $\mathrm{Ho}^+(\mathcal{I})$ quasi-isomorphic to zero into objects of $\mathrm{Ho}^+(\mathcal{B})$ satisfying the same property. Therefore, $f_*\colon \mathrm{Ho}^+(\mathcal{I}) \to \mathrm{Ho}^+(\mathcal{B})$ factors through $\mathrm{Ho}^+(\mathcal{I})/\mathcal{N}'$. Let $\mathbf{i}\colon \mathrm{Ho}^+(\mathcal{I})/\mathcal{N}' \xrightarrow{\sim} D^+(\mathcal{A})$ be the equivalence inverse to the one described in Prop. 3.16 (i.e., if $X \in \mathrm{D}^+(\mathcal{A})$, then $\mathbf{i}X$ is a resolution of X by objects in $\mathcal{I}$). The *right derived functor* $\mathbf{R}f_*\colon \mathrm{D}^+(\mathcal{A}) \to \mathrm{D}^+(\mathcal{B})$ is defined to be the composition

$$\mathrm{D}^+(\mathcal{A}) \xrightarrow{\mathbf{i}} \mathrm{Ho}^+(\mathcal{I})/\mathcal{N}' \xrightarrow{f_*} \mathrm{Ho}^+(\mathcal{B}) \xrightarrow{\text{quot}} \mathrm{D}^+(\mathcal{B}). \tag{3.1}$$

The derived functor $\mathbf{R}f_*$ is unique up to canonical isomorphism, in particular it does not depend on the choice of the f_*-injective subcategory $\mathcal{I}$ (see [KaSc06, Prop. 13.3.5]).

Proposition 3.19. *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be abelian categories. Let $f_*\colon \mathcal{A} \to \mathcal{B}$ and $g_*\colon \mathcal{B} \to \mathcal{C}$ be left exact functors. Assume that there exist full additive subcategories $\mathcal{I} \subseteq \mathcal{A}$ and $\mathcal{I}' \subseteq \mathcal{B}$ such that $\mathcal{I}$ is f_*-injective, $\mathcal{I}'$ is g_*-injective and $f_*\mathcal{I} \subseteq \mathcal{I}'$. Then $\mathcal{I}$ is g_*f_*-injective and induces an isomorphism*

$$\mathbf{R}(g_*f_*) \xrightarrow{\sim} \mathbf{R}g_*\mathbf{R}f_*.$$

Proof. See [KaSc06, Prop. 13.3.13 (ii)]. □

Remark 3.20. Similar statements apply to right exact functors. See [KaSc06, Remark 13.3.14].

4. Functorial cones

4.1. **Motivation.** Let $\mathcal{T}$ and $\mathcal{T}'$ be triangulated categories. Let $F, G\colon \mathcal{T} \to \mathcal{T}'$ be triangulated functors and let $\phi\colon F \to G$ be a natural transformation. One would like to define a 'cone functor' H of the morphism $F \xrightarrow{\phi} G$. More precisely, $Z\colon \mathcal{T} \to \mathcal{T}$ should be a triangulated functor and it should be equipped with natural transformations $\phi'\colon G \to H$ and $\phi''\colon H \to F[1]$ such that $F(X) \xrightarrow{\phi_X} G(X) \xrightarrow{\phi'_X} H(X) \xrightarrow{\phi''_X} F[1]$ is a distinguished triangle for each $X \in \mathcal{T}$. Unfortunately, in such a general setting it is not possible to construct such an H. The obstruction is the 'non-functoriality' of cones in triangulated categories (recall that, in general, cones are unique only up to *non-canonical* isomorphism, see [GeMa, Ch. IV §1.7] for a discussion of this).

Regardless, we would like to identify a suitable 2-category satisfying the following properties: the 'usual triangulated categories' (homotopy categories of complexes, derived categories) are contained as objects of this 2-category; all 'natural functors' (derived functors etc.) are 1-morphisms; natural transformations are 2-morphisms; the 1-morphisms have a natural triangulated structure in the sense of the previous paragraph; composition and adjoints of 1-morphisms are compatible with the triangulated structure in a suitable way.

There are several approaches to this in the literature such as DG-categories (see [Dri]) and their cousins A_∞-categories (see [Kon]). However, these theories are still in development and the literature on these is technically demanding. In this section I will take a more pedestrian approach. I construct a 2-category $\mathfrak{K}$ using the classical language of complexes etc. This 2-category provides a framework, adequate for our purposes, to deal with functors between homotopy/derived categories. The ideas are drawn from several sources: [BK], [BN], [CW], [Ro, §2.1.1] and [ST, §4]).

4.2. **Complexes of functors.** Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be additive categories. Let $F \in \mathrm{Comp}(\mathscr{H}om(\mathcal{B}, \mathcal{C}))$, $G \in \mathrm{Comp}(\mathscr{H}om(\mathcal{A}, \mathcal{B}))$. Define $F \star G \in \mathrm{Comp}(\mathscr{H}om(\mathcal{A}, \mathcal{C}))$ to be the complex whose degree n component is $\bigoplus_{i+j=n} F^i G^j$ with differential

$$d_{F\star G}\colon F^iG^j \to F^{i+1}G^j \oplus F^iG^{j+1}, \qquad d_{F\star G} = d_F \mathbb{1}_{G^j} + (-1)^i \mathbb{1}_{F^i} d_G.$$

Remark 4.1. $F \star G$ is the total complex of the double complex $\{F^iG^j\}_{i,j}$, see [KaSc06, §11.5].

Proposition 4.2. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be additive categories and let $F \in \mathrm{Comp}(\mathscr{H}om(\mathcal{C}, \mathcal{D}))$, $G \in \mathrm{Comp}(\mathscr{H}om(\mathcal{B}, \mathcal{C}))$, $H \in \mathrm{Comp}(\mathscr{H}om(\mathcal{A}, \mathcal{B}))$. Then*

$$(F \star G) \star H = F \star (G \star H).$$

Proof. The degree n component of both $(F \star G) \star H$ and $F \star (G \star H)$ is $\bigoplus_{i+j+k=n} F^iG^jH^k$. It remains to check that the differentials on both sides coincide. The differential for $(F \star G) \star H$, $d_{(F\star G)\star H}\colon F^iG^jH^k \to F^{i+1}G^jH^k \oplus F^iG^{j+1}H^k \oplus F^iG^jH^{k+1}$ is

$$\begin{aligned} d_{(F\star G)\star H} &= d_{F\star G}\mathbb{1}_{H^k} + (-1)^{i+j}\mathbb{1}_{F^iG^j}d_H \\ &= d_F\mathbb{1}_{G^jH^k} + (-1)^i\mathbb{1}_{F^i}d_G\mathbb{1}_{H^k} + (-1)^{i+j}\mathbb{1}_{F^iG^j}d_H. \end{aligned}$$

The differential for $F \star (G \star H)$, $d_{F\star(G\star H)}\colon F^iG^jH^k \to F^{i+1}G^jH^k \oplus F^iG^{j+1}H^k \oplus F^iG^jH^{k+1}$ is

$$\begin{aligned} d_{F\star(G\star H)} &= d_F\mathbb{1}_{G^jH^k} + (-1)^i\mathbb{1}_{F^i}d_{G\star H} \\ &= d_F\mathbb{1}_{G^jH^k} + (-1)^i\mathbb{1}_{F^i}d_G\mathbb{1}_{H^k} + (-1)^{i+j}\mathbb{1}_{F^iG^j}d_H. \end{aligned}$$ □

4.3. **Transposes of complexes of functors.** Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories. Let (f_i^*, f_{i*}), $i \in \mathbb{Z}$, be adjunctions between functors $f_{i*}\colon \mathcal{A} \to \mathcal{B}$ and $f_i^*\colon \mathcal{B} \to \mathcal{A}$. Suppose we have a complex of functors

$$F_* = \cdots \xrightarrow{d_{-2}} f_{-1*} \xrightarrow{d_{-1}} f_{0*} \xrightarrow{d_0} f_{1*} \xrightarrow{d_1} \cdots,$$

with f_{0*} in degree 0. Set

$$F^* = \cdots \xrightarrow{d_1^\vee} f_1^* \xrightarrow{d_0^\vee} f_0^* \xrightarrow{d_{-1}^\vee} f_{-1}^* \xrightarrow{d_{-2}^\vee} \cdots,$$

with f_0^* in degree 0. Then Prop. 2.3 (iii) and Prop. 2.4 (ii) imply that F^* is also a complex. The degree 0 term of $F^* \star F$ is $\bigoplus_{i\in\mathbb{Z}} f_i^* f_{i*}$. View the identity functor as a complex concentrated in degree 0. Define $\mathrm{ev}\colon F^*F_* \to \mathrm{id}$ by

$$\begin{pmatrix} \cdots & -\varepsilon_{-2} & -\varepsilon_{-1} & \varepsilon_0 & \varepsilon_1 & -\varepsilon_2 & -\varepsilon_3 & \varepsilon_4 & \varepsilon_5 & \cdots \end{pmatrix} : \bigoplus_{i\in\mathbb{Z}} f_i^* f_{i*} \to \mathrm{id},$$

where ε_i is the counit of the adjunction (f_i^*, f_{i*}). The differential on the degree -1 term of $F^* \star F_*$ is given by

$$\begin{pmatrix} d_i^\vee \mathbb{1}_{f_{i*}} \\ (-1)^{i+1}\mathbb{1}_{f_{i+1}^*} d_i \end{pmatrix} : f_{i+1}^* f_{i*} \to f_i^* f_{i*} \oplus f_{i+1}^* f_{i+1*}.$$

This combined with Prop. 2.3 (i) implies that ev is a chain map. Similarly, the degree 0 term of $F_* \star F^*$ is $\bigoplus_{i\in\mathbb{Z}} f_{i*} f_i^*$. Define $\mathrm{coev}\colon \mathrm{id} \to F_* \star F^*$ by

$$\begin{pmatrix} \vdots \\ -\eta_{-2} \\ -\eta_{-1} \\ \eta_0 \\ \eta_1 \\ -\eta_2 \\ -\eta_3 \\ \eta_4 \\ \eta_5 \\ \vdots \end{pmatrix} : \mathrm{id} \to \bigoplus_{i\in\mathbb{Z}} f_{i*} f_i^*, .$$

where η_i is the counit of the adjunction (f_i^*, f_{i*}). The differential on the degree 0 term is given by

$$\begin{pmatrix} d_i \mathbb{1}_{f_i^*} \\ (-1)^i \mathbb{1}_{f_{i*}} d_{i-1}^\vee \end{pmatrix} : f_{i*} f_i^* \to f_{i+1*} f_i^* \oplus f_{i*} f_{i-1}^*.$$

This combined with Prop. 2.3 (i) gives that coev is a chain map.

Proposition 4.3. *The compositions*

$$F_* \xrightarrow{\mathrm{coev}\mathbb{1}_{F_*}} F_* \star F^* \star F_* \xrightarrow{\mathbb{1}_{F_*}\mathrm{ev}} F_* \qquad \text{and} \qquad F^* \xrightarrow{\mathbb{1}_{F^*}\mathrm{coev}} F^* \star F_* \star F^* \xrightarrow{\mathrm{ev}\mathbb{1}_{F^*}} F^*$$

are equal to the identity on F_* and F^*, respectively.

Proof. This follows from the corresponding properties of η_i and ε_i. See Example 5.5. □

Let (f_i^*, f_{i*}) and (g_i^*, g_{i*}), $i \in \mathbb{Z}$, be sequences of adjunctions. Let

$$F_* = \cdots \to f_{-1*} \xrightarrow{d_{-1}} f_{0*} \xrightarrow{d_0} \cdots \qquad \text{and} \qquad G_* = \cdots \to g_{-1*} \xrightarrow{d'_{-1}} g_{0*} \xrightarrow{d'_0} \cdots$$

be complexes of functors with f_{0*} and g_{0*} in degree 0. Set

$$F^* = \cdots \to f_1^* \xrightarrow{d_0^\vee} f_0^* \xrightarrow{d_{-1}^\vee} \cdots \qquad \text{and} \qquad G^* = \cdots \to g_1^* \xrightarrow{d_0'^\vee} g_0^* \xrightarrow{d_{-1}'^\vee} \cdots$$

with f_0^* and g_0^* in degree 0. Let $\phi = \{\phi_i \colon f_{i*} \to g_{i*}\}$ be a chain map $F_* \to G_*$. Define $\phi^\vee \colon G^* \to F^*$ by $\phi^\vee = \{\phi_i^\vee \colon g_i^* \to f_i^*\}$ (see (2.2)). Let H_* be the cone of ϕ (see (1.1)). By definition, the degree i component of H_* is $g_{i*} \oplus f_{i+1*}$, with differential $g_{i*} \oplus f_{i+1*} \xrightarrow{\begin{pmatrix} d'_i & \phi_{i+1} \\ 0 & -d_{i+1} \end{pmatrix}} g_{i+1*} \oplus f_{i+2*}$. Let H^* be the complex with $f^*_{-i+1} \oplus g^*_{-i}$ in degree i and differential $f^*_{-i+1} \oplus g^*_{-i} \xrightarrow{\begin{pmatrix} -d^\vee_{-i} & \phi^\vee_{-i} \\ 0 & d'^\vee_{-i-1} \end{pmatrix}} f^*_{-i} \oplus g^*_{-i-1}$. According to Prop. 4.3, H^* is 'left adjoint' to H_*. Let (H^*, H_*) be the 'adjunction' given by Prop. 4.3. Let $F_* \xrightarrow{\phi} G_* \xrightarrow{\iota} H_* \xrightarrow{\delta} F_*[1]$ be the standard triangle (see (1.2)). Taking transposes (2.2) we get a sequence of maps $F^*[-1] \xrightarrow{\delta^\vee} H^* \xrightarrow{\iota^\vee} G^* \xrightarrow{\phi^\vee} F^*$. By Prop. 2.4 (i), $\delta^\vee$ is given by $f^*_{-i+1} \xrightarrow{\begin{pmatrix} \mathrm{id} \\ 0 \end{pmatrix}} f^*_{-i+1} \oplus g^*_{-i}$ in degree i. Similarly, $\iota^\vee$ is given by $f^*_{-i+1} \oplus g^*_{-i} \xrightarrow{(0\ \mathrm{id})} g^*_{-i}$ in degree i. Let H' denote the cone of $G^* \xrightarrow{\phi^\vee} F^*$. Let $G^* \xrightarrow{\phi^\vee} F^* \xrightarrow{\iota'} H' \xrightarrow{\delta'} G^*[1]$ be the corresponding standard triangle. The degree i component of H' is $f^*_{-i} \oplus g^*_{-i-1}$, with differential $f^*_{-i} \oplus g^*_{-i-1} \xrightarrow{\begin{pmatrix} d^\vee_{-i-1} & \phi^\vee_{-i-1} \\ 0 & -d'^\vee_{-i-2} \end{pmatrix}} f^*_{-i-1} \oplus g^*_{-i-2}$. So the degree i component of $H'[-1]$ is $f^*_{-i+1} \oplus g^*_{-i}$, with differential $f^*_{-i+1} \oplus g^*_{-i} \xrightarrow{\begin{pmatrix} -d^\vee_{-i} & -\phi^\vee_{-i} \\ 0 & d'^\vee_{-i-1} \end{pmatrix}} f^*_{-i} \oplus g^*_{-i-1}$. Let $\psi \colon H^* \to H'[-1]$ be the chain map given by $f^*_{-i+1} \oplus g^*_{-i} \xrightarrow{\begin{pmatrix} \mathrm{id} & 0 \\ 0 & -\mathrm{id} \end{pmatrix}} f^*_{i+1} \oplus g^*_{-i}$ in degree i It is clear that ψ defines an isomorphism of complexes $H^* \simeq H'[-1]$. Further, the diagram

$$\begin{array}{ccccccc} F^*[-1] & \xrightarrow{-\delta^\vee} & H^* & \xrightarrow{\iota^\vee} & G^* & \xrightarrow{\phi^\vee} & F^* \\ \| & & \psi \downarrow \wr & & \| & & \| \\ F^*[-1] & \xrightarrow{-\iota'[-1]} & H'[-1] & \xrightarrow{-\delta'[-1]} & G^* & \xrightarrow{\phi^\vee} & F^* \end{array} \tag{4.1}$$

commutes. Indeed, in degree i this diagram is

$$\begin{array}{ccccccc}
f^*_{-i-1} & \xrightarrow{\left(\begin{smallmatrix}-\mathrm{id}\\ 0\end{smallmatrix}\right)} & f^*_{-i+1} \oplus g^*_{-i} & \xrightarrow{(0\ \mathrm{id})} & g^*_{-i} & \xrightarrow{\phi_i^\vee} & f^*_{-i} \\
\Vert & & \downarrow{\scriptstyle \left(\begin{smallmatrix}\mathrm{id} & 0\\ 0 & -\mathrm{id}\end{smallmatrix}\right)} & & \Vert & & \Vert \\
f^*_{-i-1} & \xrightarrow{\left(\begin{smallmatrix}-\mathrm{id}\\ 0\end{smallmatrix}\right)} & f^*_{-i+1} \oplus g^*_{-i} & \xrightarrow{(0\ -\mathrm{id})} & g^*_{-i} & \xrightarrow{\phi_i^\vee} & f^*_{-i}
\end{array}$$

4.4. **A general setting.** Define a 2-category $\mathfrak{C}$ as follows:

(i) Objects: additive categories;

(ii) 1-morphisms: complexes of functors from $\mathcal{A}$ to $\mathcal{B}$, with composition defined via the $\star$ operation ($\star$ is associative by Prop. 4.2);

(iii) 2-morphisms: morphisms in $\mathrm{Ho}(\mathscr{H}om(\mathcal{A},\mathcal{B}))$, i.e., homotopy classes of chain maps between complexes of functors.

Let $\mathfrak{T}$ be the 2-category of triangulated categories (objects: triangulated categories, 1-morphisms: exact functors, 2-morphism: natural transformations of exact functors). Define a 2-functor $\Phi\colon \mathfrak{C} \to \mathfrak{T}$ as follows: Φ sends an additive category $\mathcal{A}$ to its homotopy category $\mathrm{Ho}(\mathcal{A})$. If F is a complex of functors from $\mathcal{A}$ to $\mathcal{B}$ and $X \in \mathrm{Ho}(\mathcal{A})$ set $\Phi(F)(X)$ to be the complex whose degree n component is $\bigoplus_{i+j=n} F^i(X^j)$ with differential $d\colon F^iX^j \to F^{i+1}(X^j) \oplus F^i(X^{j+1})$ given by $d = d_F \mathbb{1}_{X^j} + (-1)^i \mathbb{1}_{F^i} d_X$.

Let $\mathcal{N}_{\mathcal{A}\mathcal{B}}$ be the full subcategory of $\mathscr{H}om_{\mathfrak{C}}(\mathcal{A},\mathcal{B})$ consisting of those complexes of functors that are sent to the zero functor $\mathrm{Ho}(\mathcal{A}) \to \mathrm{Ho}(\mathcal{B})$ by Φ.

Proposition 4.4. *$\mathcal{N}_{\mathcal{A}\mathcal{B}}$ is a localizing subcategory of $\mathscr{H}om_{\mathfrak{C}}(\mathcal{A},\mathcal{B})$.*

Proof. This is straightforward. □

Define the 2-category $\mathfrak{K}$ as follows.

(i) Objects: $\mathrm{Ho}(\mathcal{A})$, where $\mathcal{A}$ is an additive category;

(ii) 1-morphisms: $\mathscr{H}om_{\mathfrak{K}}(\mathrm{Ho}(\mathcal{A}),\mathrm{Ho}(\mathcal{B}))$ = complexes of functors $\mathcal{A} \to \mathcal{B}$;

(iii) 2-morphisms: $\mathrm{Hom}_{\mathfrak{C}}(F,G)/\mathcal{N}_{\mathcal{A}\mathcal{B}}-\mathrm{qis}$, for $F,G \in \mathscr{H}om_{\mathfrak{K}}(\mathrm{Ho}(\mathcal{A}),\mathrm{Ho}(\mathcal{B}))$;

It is clear that the 2-functor $\Phi\colon \mathfrak{C} \to \mathfrak{T}$ factors through $\mathfrak{K}$. To ease notation we will write FG instead of $F \star G$ for composition of 1-morphisms in $\mathfrak{K}$. Further, for $F \in \mathscr{H}om_{\mathfrak{K}}(\mathcal{T}_1,\mathcal{T}_2)$ and $X \in \mathcal{T}_1$, write $F(X)$ instead of $\Phi(F)(X)$.

Theorem 4.5. *Let $\mathcal{T}_1,\mathcal{T}_2 \in \mathfrak{K}$, then $\mathscr{H}om_{\mathfrak{K}}(\mathcal{T}_1,\mathcal{T}_2)$ is triangulated. Furthermore:*

(i) *for each $\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3 \in \mathfrak{K}$, each $G \in \mathscr{H}om_{\mathfrak{K}}(\mathcal{T}_1,\mathcal{T}_2)$ and each $F \in \mathscr{H}om_{\mathfrak{K}}(\mathcal{T}_2,\mathcal{T}_3)$, we have*

$$F[1]G = (FG)[1] = FG[1];$$

(ii) *let $\mathcal{T}_i \in \mathfrak{K}$, $i \in \{0,1,2,3\}$. Let $F,G,H \in \mathscr{H}om_{\mathfrak{K}}(\mathcal{T}_1,\mathcal{T}_2)$, $E \in \mathscr{H}om_{\mathfrak{K}}(\mathcal{T}_2,\mathcal{T}_3)$ and let $E' \in \mathscr{H}om_{\mathfrak{K}}(\mathcal{T}_0,\mathcal{T}_1)$. If $F \xrightarrow{f} G \xrightarrow{g} H \xrightarrow{h} F[1]$ is a distinguished triangle, then both*

$$EF \xrightarrow{\mathbb{1}_E f} EG \xrightarrow{e_E g} EH \xrightarrow{\mathbb{1}_E h} EF[1] \qquad \textit{and} \qquad FE' \xrightarrow{f \mathbb{1}_{E'}} GE' \xrightarrow{g \mathbb{1}_{E'}} HE \xrightarrow{h \mathbb{1}_{E'}} FE'[1]$$

are distinguished triangles;

(iii) *let* $F_*, G_* \in \mathscr{H}om_{\mathfrak{K}}(\mathcal{T}_1, \mathcal{T}_2)$ *and let* $F^*, G^* \in \mathscr{H}om_{\mathfrak{K}}(\mathcal{T}_2, \mathcal{T}_1)$. *Let* (F^*, F_*) *and* (G^*, G_*) *be adjunctions. Let* $\phi \in \mathrm{Hom}_{\mathfrak{K}}(F_*, G_*)$. *Let* $F_* \xrightarrow{\phi} G_* \xrightarrow{\phi'} H_* \xrightarrow{\phi''} F_*[1]$ *be a distinguished triangle. Then there exists* $H^* \in \mathscr{H}om_{\mathfrak{K}}(\mathcal{T}_2, \mathcal{T}_1)$ *and an adjunction* (H^*, H_*) *such that* $H^* \xrightarrow{\phi'^\vee} G^* \xrightarrow{\phi^\vee} F^* \xrightarrow{\phi''^\vee[1]} H^*[1]$ *is a distinguished triangle;*

Proof. (i) and (ii) are clear. (iii) is the content of (4.1). □

It is clear how to modify the definition of $\mathfrak{K}$ to incorporate various boundedness conditions on complexes. Further, since derived functors factor through various homotopy categories (3.1), it is also clear that we may handle these in the framework of $\mathfrak{K}$.

5. Equivalences

In §1 and §2 of this section some standard 'dévissage' type results on isomorphisms between functors are presented. In §3 these results, combined with the formalism of the previous sections, culminate in Thm. 5.4. Although the proof of Thm. 5.4 is completely formal and quite trivial, these trivialities seem to have got us somewhere. Thm. 5.4 outlines a situation in which a symmetry at the level of Grothendieck groups can be lifted to an auto-equivalence of the derived category. Indeed, there are various conjectures in the literature (especially in the modular representation theory of finite groups, see [RiICM]) on the existence of derived equivalences that are based on the evidence of symmetries at the level of Grothendieck groups.

A few comments on the 'originality' of Thm. 5.4: this result is an abstraction of [Ri, Thm. 2.1]. The basic idea can also be found in [Ro, §2.2.3], implicitly in [ABG, Lemma 4.1.1] and implicitly in [Vo, Thm. 7.3.16].

Since a special case of Thm. 5.4 is all that is needed for the analysis in the sequel, I have included a second 'proof from scratch' for this special case in the form of Example 5.5. In fact, [Ri, Thm. 2.1] along with the computations contained in Example 5.5 are what originally led me to Thm. 5.4.

5.1. **Equivalences in abelian categories.**

Lemma 5.1. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be a abelian categories such that every object in* $\mathcal{A}$ *has finite length. Let* $f, g\colon \mathcal{A} \to \mathcal{B}$ *be exact functors. Suppose* $\varepsilon\colon f \to g$ *is a natural transformation that is an isomorphism on simple objects, i.e.,* $\varepsilon_L\colon fL \xrightarrow{\sim} gL$ *for each simple object* $L \in \mathcal{A}$. *Then* $\varepsilon\colon f \to g$ *is an isomorphism.*

Proof. We need to show that $\varepsilon_X\colon fX \xrightarrow{\sim} gX$ for each $X \in \mathcal{A}$. Proceed by induction on the length of X. The base case is provided by the statement for simple objects. Assume that the statement is true for objects of length $< n$ and suppose X is of length n. Then we have an exact sequence $0 \to X' \to X \to L \to 0$, with X' of length $n-1$ and L simple. This gives a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & fX' & \longrightarrow & fX & \longrightarrow & fL & \longrightarrow & 0 \\ & & {\scriptstyle\varepsilon_{X'}}\downarrow{\scriptstyle\sim} & & {\scriptstyle\varepsilon_X}\downarrow & & {\scriptstyle\varepsilon_L}\downarrow{\scriptstyle\sim} & & \\ 0 & \longrightarrow & gX' & \longrightarrow & gX & \longrightarrow & gL & \longrightarrow & 0 \end{array}$$

The outer vertical arrows are isomorphisms by the induction hypothesis. So the Five Lemma [KaSc06, Lemma 8.3.13] forces the middle arrow to also be an isomorphism. ☐

Proposition 5.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories such that every object in $\mathcal{A}$ and $\mathcal{B}$ has finite length. Let (f^*, f_*) be an adjunction between exact functors $f_*\colon \mathcal{A} \to \mathcal{B}$ and $f^*\colon \mathcal{B} \to \mathcal{A}$. Then f^* and f_* are mutually inverse equivalences if and only if f^* and f_* induce mutually inverse operators at the level of Grothendieck groups.*

Proof. It is clear that if f^* and f_* are mutually inverse equivalences, then they induce mutually inverse operators at the level of Grothendieck groups. Conversely, let $\varepsilon\colon f^*f_* \to \mathrm{id}_{\mathcal{A}}$ be the unit of adjunction. Lets prove that ε is an isomorphism. By the previous lemma, it suffices to show that $\varepsilon\colon f^*f_*L \xrightarrow{\sim} L$ for each simple object $L \in \mathcal{A}$. As $[f^*f_*L] = [L]$, we infer that $f^*f_*L \simeq L$. Further, by Lemma 2.1, $\varepsilon\colon f^*f_*L \to L$ is non-zero, forcing it to be an isomorphism. Now let η be the counit of adjunction. A similar argument shows that $\eta\colon \mathrm{id}_{\mathcal{B}} \to f_*f^*$ is also an isomorphism. ☐

5.2. **Equivalences in triangulated categories.** We ask the reader to recall the notion of filtrations in triangulated categories and some of the notation in Ch. 3 §3.1. In particular, if $\mathcal{L}$ is a subcategory of a triangulated category $\mathcal{T}$, then

$$\mathcal{L} * \mathcal{L} = \{Y \in \mathcal{T} \mid \text{there exists } X \to Y \to Z \rightsquigarrow, \text{ with } [X], [Z] \in \mathcal{L}\},$$

where $[X]$ denotes the isomorphism class of X. Further, recall that $\mathcal{L}^{*i} = \mathcal{L} * \mathcal{L}^{*i-1}$, and that $\mathcal{L}^{*\infty} = \bigcup_{i \in \mathbb{Z}_{\geq 0}} \mathcal{L}^{*i}$.

Lemma 5.3. *Let $\mathcal{T}$ and $\mathcal{T}'$ be triangulated categories. Let $\mathcal{L} \subset \mathcal{T}$ be a subcategory (not necessarily triangulated). Suppose that $\mathcal{L}^{*\infty} = \mathcal{T}$. Let $f, g\colon \mathcal{T} \to \mathcal{T}'$ be exact functors and let $\varepsilon\colon f \to g$ be a morphism of exact functors. If $\varepsilon_L\colon fL \to gL$ is an isomorphism for each $L \in \mathcal{L}$, then $\varepsilon\colon f \to g$ is an isomorphism.*

Proof. Proceed by induction, assume that if $i < n$, then $\varepsilon_L\colon fL \to gL$ is an isomorphism for each $L \in \mathcal{L}^{*i}$. Let $M \in \mathcal{L}^{*n}$, then we have a distinguished triangle $N \to M \to L \rightsquigarrow$ with $N \in \mathcal{L}^{*n-1}$ and $L \in \mathcal{L}$. So we obtain a commutative diagram

$$\begin{array}{ccccc} N & \longrightarrow & M & \longrightarrow & L \rightsquigarrow \\ {\scriptstyle \varepsilon_N}\downarrow{\scriptstyle \sim} & & {\scriptstyle \varepsilon_M}\downarrow & & {\scriptstyle \varepsilon_L}\downarrow{\scriptstyle \sim} \\ N & \longrightarrow & M & \longrightarrow & L \rightsquigarrow \end{array}$$

The outer vertical arrows are isomorphisms by hypothesis. This forces the middle arrow to also be an isomorphism [KaSc06, Prop. 10.1.15]. ☐

5.3. **Equivalences in derived categories.** In a moment we will be analyzing some functors between derived categories that are defined via complexes of exact functors between the underlying abelian categories. We remind the reader that this is in the 2-categorical setting of Ch. 4. In particular, composition of such complexes of functors is given by the $\star$ operation of Ch. 2 §4.2. We also urge the reader to reacquaint themselves with the language of adjunctions as outlined in Ch. 2 §2.1.

The following is an abstraction of [Ri, Thm. 2.1], in a slightly different form it is also contained in [Ro, §2.2.3] (also see [ABG, Lemma 4.1.1] and [Vo, Thm. 7.3.16].

Theorem 5.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Assume each object in $\mathcal{A}$ has finite length. Let (π^*, π_*) and $(\pi_*, \pi^!)$ be adjunctions between exact functors $\pi_*\colon \mathcal{A} \to \mathcal{B}$ and $\pi^*, \pi^!\colon \mathcal{B} \to \mathcal{A}$. Then we have the data of four morphisms (units and counits):*

$$\eta\colon \mathrm{id}_{\mathcal{B}} \to \pi_*\pi^*, \qquad \varepsilon\colon \pi^*\pi_* \to \mathrm{id}_{\mathcal{A}}, \qquad \eta'\colon \mathrm{id}_{\mathcal{A}} \to \pi^!\pi_*, \qquad \varepsilon'\colon \pi_*\pi^! \to \mathrm{id}_{\mathcal{B}}.$$

Define complexes of functors Θ^* and $\Theta^!$:

$$\Theta^* = 0 \to \pi^*\pi_* \xrightarrow{\varepsilon} \mathrm{id}_{\mathcal{A}} \to 0 \qquad \text{and} \qquad \Theta^! = 0 \to \mathrm{id}_{\mathcal{A}} \xrightarrow{\eta'} \pi^!\pi_* \to 0,$$

with $\pi^*\pi_*$ and $\pi^!\pi_*$ in degree 0. By Lemma 2.5 and Prop. 4.3, Θ^* is left adjoint to $\Theta^!$. Fix an adjunction $(\Theta^*, \Theta^!)$ and denote the unit by coev *and the counit by* ev*.*

(i) *If $[\pi^*\pi_*\pi^!\pi_*X] = [\pi^*\pi_*X] + [\pi^!\pi_*X]$ in $K_0(\mathcal{A})$ for each $X \in \mathcal{A}$, then $\mathrm{ev}\colon \Theta^*\Theta^! \to \mathrm{id}$ is an isomorphism of functors on $\mathrm{D^b}(\mathcal{A})$.*

(ii) *If $[\pi^!\pi_*\pi^*\pi_*X] = [\pi^*\pi_*X] + [\pi^!\pi_*X]$ in $K_0(\mathcal{A})$ for each $X \in \mathcal{A}$, then $\mathrm{coev}\colon \mathrm{id} \to \Theta^!\Theta^*$ is an isomorphism of functors on $\mathrm{D^b}(\mathcal{A})$.*

Proof. By definition (see Ch. 2 §4.2 and Ch. 4), the functor $\Theta^*\Theta^!$ is given by the complex

$$0 \longrightarrow \pi^*\pi_* \xrightarrow{\begin{pmatrix} \varepsilon \\ \mathbb{1}_{\pi^*\pi_*}\eta' \end{pmatrix}} \mathrm{id}_{\mathcal{A}} \oplus \pi^*\pi_*\pi^!\pi_* \xrightarrow{\begin{pmatrix} -\eta' & \varepsilon\mathbb{1}_{\pi^!\pi_*} \end{pmatrix}} \pi^!\pi_* \longrightarrow 0.$$

By definition of the unit η' and the counit ε' (see Ch. 2 §2.1), the composition

$$\pi^*\pi_* \xrightarrow{\mathbb{1}_{\pi^*\pi_*}\eta'} \pi^*\pi_*\pi^!\pi_* \xrightarrow{\mathbb{1}_{\pi^*}\varepsilon'\mathbb{1}_{\pi_*}} \pi^*\pi_*$$

is the identity on $\pi^*\pi_*$. Thus, $\pi^*\pi_* \xrightarrow{\begin{pmatrix} \varepsilon \\ \mathbb{1}_{\pi^*\pi_*}\eta' \end{pmatrix}} \mathrm{id}_{\mathcal{A}} \oplus \pi^*\pi_*\pi^!\pi_*$ is a monomorphism. Similarly, $\mathrm{id}_{\mathcal{A}} \oplus \pi^*\pi_*\pi^!\pi_* \xrightarrow{\begin{pmatrix} -\eta' & \varepsilon\mathbb{1}_{\pi^!\pi_*} \end{pmatrix}} \pi^!\pi_*$ is an epimorphism. Hence, if $X \in \mathcal{A}$, then $\Theta^*\Theta^!X$ is isomorphic (in $\mathrm{D^b}(\mathcal{A})$) to an object in $\mathcal{A}$. Let $L \in \mathcal{A}$ be simple, then under the hypothesis of (i):

$$[\Theta^*\Theta^!L] = [\pi^*\pi_*\pi^!\pi_*L] + [L] - [\pi^*\pi_*L] - [\pi^!\pi_*L] = [L] \quad \text{in } K_0(\mathcal{A}).$$

This forces $\Theta^*\Theta^!L \simeq L$. Lemma 2.1 (ii) gives that $\mathrm{ev}\colon \Theta^*\Theta^!L \to L$ is non-zero. Since L is simple, this implies that $\mathrm{ev}\colon \Theta^*\Theta^!L \to L$ is an isomorphism. As every object in $\mathcal{A}$ is of finite length, every object in $\mathcal{A}$ is filtered by simple objects. Further, every object in $\mathrm{D^b}(\mathcal{A})$ is filtered by shifts of objects in $\mathcal{A}$. Thus, every object in $\mathrm{D^b}(\mathcal{A})$ is filtered by shifts of the simple objects in $\mathcal{A}$. Applying Lemma 5.3 now gives (i).

The functor $\Theta^!\Theta^*$ is given by the complex

$$0 \longrightarrow \pi^*\pi_* \xrightarrow{\begin{pmatrix} \eta'\mathbb{1}_{\pi^*\pi_*} \\ -\varepsilon \end{pmatrix}} \pi^!\pi_*\pi^*\pi_* \oplus \mathrm{id}_{\mathcal{A}} \xrightarrow{\begin{pmatrix} \mathbb{1}_{\pi^!\pi_*}\varepsilon & \eta' \end{pmatrix}} \pi^!\pi_* \longrightarrow 0\,.$$

Now an argument similar to the one for (i) gives (ii). ∎

Example 5.5. As promised in the introduction, we will now work out a 'proof from scratch' of Thm. 5.4 in the special case $\pi^! = \pi^*$.

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Assume each object in $\mathcal{A}$ has finite length. Let (π^*, π_*) and (π_*, π^*) be adjunctions between exact functors $\pi_* \colon \mathcal{A} \to \mathcal{B}$ and $\pi^* \colon \mathcal{B} \to \mathcal{A}$. Then we have the data of four morphisms (units and counits):

$$\eta \colon \mathrm{id}_{\mathcal{B}} \to \pi_*\pi^*, \qquad \varepsilon \colon \pi^*\pi_* \to \mathrm{id}_{\mathcal{A}}, \qquad \eta' \colon \mathrm{id}_{\mathcal{A}} \to \pi^*\pi_* \qquad \varepsilon' \colon \pi_*\pi^* \to \mathrm{id}_{\mathcal{B}}.$$

Let $\Theta^* = 0 \to \pi^*\pi_* \xrightarrow{\varepsilon} \mathrm{id}_{\mathcal{A}} \to 0$ and $\Theta^! = 0 \to \mathrm{id}_{\mathcal{A}} \xrightarrow{\eta'} \pi^*\pi_* \to 0$ with $\pi^*\pi_*$ in degree 0 in both cases. Let's show that Θ^* is left adjoint to $\Theta^!$. It is helpful to keep track of terms in this computation 'in color' (I apologize to the reader trying to read this in print). The functor $\Theta^*\Theta^!$ is given by the complex

$$0 \longrightarrow \pi^*\pi_*\mathrm{id}_{\mathcal{A}} \xrightarrow{\begin{pmatrix} \varepsilon \\ \mathbb{1}_{\pi^*\pi_*}\eta' \end{pmatrix}} \mathrm{id}_{\mathcal{A}}\mathrm{id}_{\mathcal{A}} \oplus \pi^*\pi_*\pi^*\pi_* \xrightarrow{\begin{pmatrix} -\eta' & \varepsilon\mathbb{1}_{\pi^*\pi_*} \end{pmatrix}} \mathrm{id}_{\mathcal{A}}\pi^*\pi_* \longrightarrow 0$$

with $\mathrm{id}_{\mathcal{A}}\mathrm{id}_{\mathcal{A}} \oplus \pi^*\pi_*\pi^*\pi_*$ in degree 0. Define $\mathrm{ev} \colon \Theta^*\Theta^! \to \mathrm{id}_{\mathcal{A}}$ by

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi^*\pi_*\mathrm{id}_{\mathcal{A}} & \xrightarrow{\begin{pmatrix} \varepsilon \\ \mathbb{1}_{\pi^*\pi_*}\eta' \end{pmatrix}} & \mathrm{id}_{\mathcal{A}}\mathrm{id}_{\mathcal{A}} \oplus \pi^*\pi_*\pi^*\pi_* & \xrightarrow{\begin{pmatrix} -\eta' & \varepsilon\mathbb{1}_{\pi^*\pi_*} \end{pmatrix}} & \mathrm{id}_{\mathcal{A}}\pi^*\pi_* & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow {\scriptstyle \begin{pmatrix} -\mathrm{id} & \varepsilon\circ\mathbb{1}_{\pi^*}\varepsilon'\mathbb{1}_{\pi_*} \end{pmatrix}} & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & \mathrm{id}_{\mathcal{A}} & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

We have

$$\begin{pmatrix} -\mathrm{id} & \varepsilon\circ\mathbb{1}_{\pi^*}\varepsilon'\mathbb{1}_{\pi_*} \end{pmatrix} \circ \begin{pmatrix} \varepsilon \\ \mathbb{1}_{\pi^*\pi_*}\eta' \end{pmatrix} = -\varepsilon + \varepsilon \circ \mathbb{1}_{\pi^*}\varepsilon'\mathbb{1}_{\pi_*} \circ \mathbb{1}_{\pi^*\pi_*}\eta' = 0,$$

where the last equality is by the definition of the unit η' and the counit ε' (see Ch. 2 §2.1). Thus, ev is a chain map. The functor $\Theta^!\Theta^*$ is given by the complex

$$0 \longrightarrow \mathrm{id}_{\mathcal{A}}\pi^*\pi_* \xrightarrow{\begin{pmatrix} \eta'\mathbb{1}_{\pi^*\pi_*} \\ -\varepsilon \end{pmatrix}} \pi^*\pi_*\pi^*\pi_* \oplus \mathrm{id}_{\mathcal{A}}\mathrm{id}_{\mathcal{A}} \xrightarrow{\begin{pmatrix} \mathbb{1}_{\pi^*\pi_*}\varepsilon & \eta' \end{pmatrix}} \pi^*\pi_*\mathrm{id}_{\mathcal{A}} \longrightarrow 0$$

with $\pi^*\pi_*\pi^*\pi_* \oplus \mathrm{id}_{\mathcal{A}}\mathrm{id}_{\mathcal{A}}$ in degree 0. Define $\mathrm{coev} \colon \mathrm{id}_{\mathcal{A}} \to \Theta^!\Theta^*$ by

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \mathrm{id}_{\mathcal{A}} & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow {\scriptstyle \begin{pmatrix} \mathbb{1}_{\pi^*}\eta\mathbb{1}_{\pi_*}\circ\eta' \\ -\mathrm{id} \end{pmatrix}} & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{id}_{\mathcal{A}}\pi^*\pi_* & \xrightarrow{\begin{pmatrix} \eta'\mathbb{1}_{\pi^*\pi_*} \\ -\varepsilon \end{pmatrix}} & \pi^*\pi_*\pi^*\pi_* \oplus \mathrm{id}_{\mathcal{A}}\mathrm{id}_{\mathcal{A}} & \xrightarrow{\begin{pmatrix} \mathbb{1}_{\pi^*\pi_*}\varepsilon & \eta' \end{pmatrix}} & \pi^*\pi_*\mathrm{id}_{\mathcal{A}} & \longrightarrow & 0
\end{array}$$

We have

$$\begin{pmatrix} \mathbb{1}_{\pi^*\pi_*}\varepsilon & \eta' \end{pmatrix} \circ \begin{pmatrix} \mathbb{1}_{\pi_*}\eta\mathbb{1}_{\pi^*}\circ\eta' \\ -\mathrm{id} \end{pmatrix} = \mathbb{1}_{\pi^*\pi_*}\varepsilon \circ \mathbb{1}_{\pi^*}\eta\mathbb{1}_{\pi_*} \circ \eta' - \eta' = 0,$$

where the last equality is by the definition of the unit η and the counit ε. Thus, coev is also a chain map. The functor $\Theta^!\Theta^*\Theta^!$ is given by the complex (we omit the differential since it is no longer relevant to the discussion)

$$\begin{aligned}
0 \to & \mathrm{id}_{\mathcal{A}}\pi^*\pi_*\mathrm{id}_{\mathcal{A}} \to \mathrm{id}\pi^*\pi_*\pi^*\pi_* \oplus \pi^*\pi_*\pi^*\pi_*\mathrm{id}_{\mathcal{A}} \oplus \mathrm{id}_{\mathcal{A}}\mathrm{id}_{\mathcal{A}}\mathrm{id}_{\mathcal{A}} \to \\
& \to \pi^*\pi_*\pi^*\pi_*\pi^*\pi_* \oplus \mathrm{id}_{\mathcal{A}}\mathrm{id}_{\mathcal{A}}\pi^*\pi_* \oplus \pi^*\pi_*\mathrm{id}_{\mathcal{A}}\mathrm{id}_{\mathcal{A}} \to \pi^*\pi_*\mathrm{id}_{\mathcal{A}}\pi^*\pi_* \to 0
\end{aligned}$$

with $\pi^*\pi_*\pi^*\pi_*\pi^*\pi_* \oplus \mathrm{id}_{\mathcal{A}}\mathrm{id}_{\mathcal{A}}\pi^*\pi_* \oplus \pi^*\pi_*\mathrm{id}_{\mathcal{A}}\mathrm{id}_{\mathcal{A}}$ in degree 0. The composition $\Theta^! \xrightarrow{\mathrm{coev}\mathbb{1}_{\Theta^!}} \Theta^!\Theta^*\Theta^! \xrightarrow{\mathbb{1}_{\Theta^!}\mathrm{ev}} \Theta^!$ is given by

$$\begin{array}{ccc}
\mathrm{id}_{\mathcal{A}} & \xrightarrow{\qquad\qquad} & \pi^*\pi_* \\
\Big\downarrow \begin{pmatrix} 0 \\ \mathbb{1}_{\pi^*}\eta\mathbb{1}_{\pi_*}\circ\eta' \\ -\mathrm{id} \end{pmatrix} & & \Big\downarrow \begin{pmatrix} \mathbb{1}_{\pi^*}\eta\mathbb{1}_{\pi_*\pi^*\pi_*}\circ\eta'\mathbb{1}_{\pi^*\pi_*} \\ -\mathbb{1}_{\pi^*\pi_*} \\ 0 \end{pmatrix} \\
\mathrm{id}_{\mathcal{A}}\pi^*\pi_*\pi^*\pi_* \oplus \pi^*\pi_*\pi^*\pi_*\mathrm{id}_{\mathcal{A}} \oplus \mathrm{id}_{\mathcal{A}}\mathrm{id}_{\mathcal{A}}\mathrm{id}_{\mathcal{A}} & \longrightarrow & \pi^*\pi_*\pi^*\pi_*\pi^*\pi_* \oplus \mathrm{id}_{\mathcal{A}}\mathrm{id}_{\mathcal{A}}\pi^*\pi_* \oplus \pi^*\pi_*\mathrm{id}_{\mathcal{A}}\mathrm{id}_{\mathcal{A}} \\
\Big\downarrow \begin{pmatrix} \varepsilon\circ\mathbb{1}_{\pi^*}\varepsilon'\mathbb{1}_{\pi^*} & 0 & -\mathrm{id} \end{pmatrix} & & \Big\downarrow \begin{pmatrix} \mathbb{1}_{\pi^*\pi_*}\varepsilon\circ\mathbb{1}_{\pi^*\pi_*\pi^*}\varepsilon'\mathbb{1}_{\pi_*} & 0 & -\mathbb{1}_{\pi^*\pi_*} \end{pmatrix} \\
\mathrm{id}_{\mathcal{A}} & \xrightarrow{\qquad\qquad} & \pi^*\pi_*
\end{array}$$

It is clear that the vertical composition on the left is the identity. Furthermore,

$$\begin{aligned}
\begin{pmatrix} \mathbb{1}_{\pi^*\pi_*}\varepsilon\circ\mathbb{1}_{\pi^*\pi_*\pi^*}\varepsilon'\mathbb{1}_{\pi_*} & 0 & -\mathbb{1}_{\pi^*\pi_*} \end{pmatrix} \circ \begin{pmatrix} \mathbb{1}_{\pi^*}\eta\mathbb{1}_{\pi_*\pi^*\pi_*}\circ\eta'\mathbb{1}_{\pi^*\pi_*} \\ -\mathbb{1}_{\pi^*\pi_*} \\ 0 \end{pmatrix} &= \mathbb{1}_{\pi^*\pi_*}\varepsilon\circ\mathbb{1}_{\pi^*\pi_*\pi^*}\varepsilon'\mathbb{1}_{\pi_*}\circ\mathbb{1}_{\pi^*}\eta\mathbb{1}_{\pi_*\pi^*\pi_*}\circ\eta'\mathbb{1}_{\pi^*\pi_*} \\
&= \mathbb{1}_{\pi^*\pi_*}\varepsilon\circ\mathbb{1}_{\pi^*}\eta\mathbb{1}_{\pi^*}\circ\mathbb{1}_{\pi^*}\varepsilon'\mathbb{1}_{\pi^*}\circ\eta'\mathbb{1}_{\pi^*\pi_*} \\
&= \mathbb{1}_{\pi^*\pi_*}.
\end{aligned}$$

The second equality is due to η and ε' being natural transformations. The third equality is by the definition of the units η, η' and the counits $\varepsilon, \varepsilon'$. So the vertical composition on the right is also the identity. Thus, the composition

$$\Theta^! \xrightarrow{\mathrm{coev}\mathbb{1}_{\Theta^!}} \Theta^!\Theta^*\Theta^! \xrightarrow{\mathbb{1}_{\Theta^!}\mathrm{ev}} \Theta^!$$

is the identity on $\Theta^!$. A similar computation shows that the composition

$$\Theta^* \xrightarrow{\mathbb{1}_{\Theta^*}\mathrm{coev}} \Theta^*\Theta^!\Theta^* \xrightarrow{\mathrm{ev}\mathbb{1}_{\Theta^*}} \Theta^*$$

is the identity on Θ^*. Thus, Θ^* is left adjoint to $\Theta^!$.

Now let $L \in \mathcal{A}$ be simple. Then $\Theta^*\Theta^!L$ is the complex

$$0 \longrightarrow \pi^*\pi_*L \xrightarrow{\begin{pmatrix} \varepsilon_L \\ \pi^*\pi_*(\eta'_L) \end{pmatrix}} L \oplus \pi^*\pi_*\pi^*\pi_*L \xrightarrow{\begin{pmatrix} -\eta'_L & \varepsilon_{\pi^*\pi_*L} \end{pmatrix}} \pi^*\pi_*L \longrightarrow 0$$

with $L \oplus \pi^*\pi_*\pi^*\pi_*L$ in degree 0. By definition of the unit η' and the counit ε', the composition

$$\pi^*\pi_*L \xrightarrow{\pi^*\pi_*(\eta'_L)} \pi^*\pi_*\pi^*\pi_*L \xrightarrow{\pi^*(\varepsilon'_{\pi_*L})} \pi^*\pi_*L$$

is the identity on $\pi^*\pi_*L$. Thus, $\pi^*\pi_*L \xrightarrow{\begin{pmatrix} \varepsilon_L \\ \pi^*\pi_*(\eta'_L) \end{pmatrix}} L \oplus \pi^*\pi_*\pi^*\pi_*L$ is a monomorphism. Similarly, $L \oplus \pi^*\pi_*\pi^*\pi_*L \xrightarrow{\begin{pmatrix} -\eta'_L & \varepsilon_{\pi^*\pi_*L} \end{pmatrix}} \pi^*\pi_*L$ is an epimorphism. Thus, $\Theta^*\Theta^!$ is isomorphic (in $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$) to its zeroth cohomology $H^0(\Theta^*\Theta^!L)$. Assume that $[\pi^*\pi_*\pi^*\pi_*L] = 2[\pi^*\pi_*L]$ in $K_0(\mathcal{A})$ for each simple $L \in \mathcal{A}$, then

$$[H^0(\Theta^*\Theta^!L)] = [\Theta^*\Theta^!L] = [\pi^*\pi_*\pi^*\pi_*L] + [L] - 2[\pi^*\pi_*L] = [L].$$

This forces $H^0(\Theta^*\Theta^!L)$ and hence $\Theta^*\Theta^!L$ to be isomorphic to L. Lemma 2.1 (ii) gives that $\mathrm{ev}\colon \Theta^*\Theta^!L \to L$ is non-zero. Since L is simple, this implies that $\mathrm{ev}\colon \Theta^*\Theta^!L \to L$ is an isomorphism. As every object in $\mathcal{A}$ is of finite length, every object in $\mathcal{A}$ is filtered by simple objects. Further, every object in $D^b(\mathcal{A})$ is filtered by shifts of objects in $\mathcal{A}$. Thus, every object in $D^b(\mathcal{A})$ is filtered by shifts of the simple objects in $\mathcal{A}$. Applying Lemma 5.3 now gives that $\mathrm{ev}\colon \Theta^*\Theta^! \to \mathrm{id}_{\mathcal{A}}$ is an isomorphism. A similar argument shows that $\mathrm{coev}\colon \mathrm{id}_{\mathcal{A}} \to \Theta^!\Theta^*$ is an isomorphism. Hence, Θ^* and $\Theta^!$ are mutually inverse derived-equivalences.

6. Category $\mathcal{O}$

This section contains no original results. For most of the statements herein, instead of reproducing the proofs, we simply point the reader to appropriate references in the literature. We have included proofs of those statements for which we could not find references and/or for which the treatment in the literature is not quite appropriate for our purposes.

For an introduction to semisimple Lie algebras, [Serre] is highly recommended.

6.1. **Preliminaries on Lie algebras.** Let $\mathfrak{g}$ be a finite dimensional, simple, complex Lie algebra. Fix a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$. Let $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ and let $\mathfrak{h}$ be the Cartan subalgebra corresponding to $\mathfrak{b}$. This gives a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$. The non-zero eigenvalues of $\mathfrak{h}$ acting on $\mathfrak{g}$ are the *roots* of $\mathfrak{g}$; this set is denoted R. Let $\alpha \in R$ and let $\mathfrak{g}_\alpha$ denote the α-eigenspace of $\mathfrak{g}$. Then $\mathfrak{g}_\alpha$ is one dimensional. The eigenvalues of $\mathfrak{h}$ acting on $\mathfrak{n}$ are the *positive roots* of $\mathfrak{g}$; this set is denoted R^+. The set of *simple roots* is the unique subset of R^+ such that each positive root can be expressed as linear combination of simple roots with positive integer coefficients.

Let $(\cdot|\cdot)\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be the Killing form on $\mathfrak{g}$. This form is non-degenerate and descends to a non-degenerate form on $\mathfrak{h}$. Let $\mathfrak{h}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$, then as $(\cdot|\cdot)$ is non-degenerate on $\mathfrak{h}$, the assignment $h \mapsto (\cdot|h)$ defines an isomorphism $\mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$. This gives a non-degenerate form, also denoted $(\cdot|\cdot)$, on $\mathfrak{h}^*$. For each $\alpha \in R^+$, define $\alpha^\vee \in \mathfrak{h}$ by $\langle -, \alpha^\vee \rangle = \frac{2(\cdot|\alpha)}{(\alpha|\alpha)}$. In particular, $\langle \alpha, \alpha^\vee \rangle = 2$.

Let W be the Weyl group of $\mathfrak{g}$. Then W acts faithfully on $\mathfrak{h}^*$. This action is is generated by the reflections s_α, $\alpha \in R^+$, where $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$ for $\lambda \in \mathfrak{h}^*$. In particular, W is finite. Define the *dot action* of W on $\mathfrak{h}^*$ by $w \cdot \lambda = w(\lambda + \rho) - \rho$, where $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$.

Set

$$P = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}, \text{ for all } \alpha \in R^+\},$$
$$P^+ = \{\lambda \in P \mid \langle \lambda, \alpha^\vee \rangle \geq 0, \text{ for all } \alpha \in R^+\}.$$

Let $w \in W$, then $w = s_1 \cdots s_m$, where each s_i is a reflection corresponding to a simple root. If m is minimal, then $s_1 \cdots s_m$ is a *reduced word* for w and we set $\ell(w) = m$.

Proposition 6.1.

(i) *There is a unique element $w_0 \in W$ of maximum length $|R^+|$, such that $w_0(\alpha) \in -R^+$ for all $\alpha \in R^+$. Moreover, $\ell(w_0 w) = \ell(w_0) - \ell(w)$ for all $w \in W$.*

(ii) *Let $s_\alpha \in W$ be the reflection corresponding to $\alpha \in R^+$. Let $w \in W$ be arbitrary. If $\ell(ws_\alpha) > \ell(w)$, then $w\alpha \in R^+$. Similarly, if $\ell(ws_\alpha) < \ell(w)$, then $w\alpha \in -R^+$.*

Proof. See [Hum, §1.6-1.8] or [Bou68, Ch. 4]. □

The *Bruhat order* on W is the partial order defined by requiring $w' \leq w$ if and only if w' occurs as a sub-word in some (and hence any) reduced word $s_1 \cdots s_m$ for w. Here a sub-word is a product $s_{i_1} \cdots s_{i_k}$ with $1 \leq i_1 < \cdots < i_k \leq m$.

Proposition 6.2. *Let $w, w' \in W$.*

(i) *Let $s \in W$ be a simple reflection. If $w' \leq w$, then $w's \leq w$ or $w's \leq ws$ (or both).*

(ii) *If $w' < w$, then $\ell(w') < \ell(w)$.*

(iii) *If $w' < w$, then there exists a sequence of roots $\alpha_1, \cdots, \alpha_m \in R^+$, such that $w = s_{\alpha_m} \cdots s_{\alpha_1} w'$, with $\ell(s_{\alpha_i} \cdots s_{\alpha_1} w') < \ell(s_{\alpha_{i+1}} s_{\alpha_i} \cdots s_{\alpha_1} w')$ for each i, where s_{α_i} is the reflection corresponding to α_i.*

(iv) *Adjacent elements in the Bruhat order differ in length by* 1.

Proof. See [Hum, §5.9-5.11]. □

6.2. **The enveloping algebra.** Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$ (not necessarily finite dimesnional or simple). The universal enveloping algebra $U(\mathfrak{g})$ is the associative algebra (with 1) generated by the vector space $\mathfrak{g}$, subject to the relations $xy - yx = [x, y]$, $x, y \in \mathfrak{g}$. Then the data of a $\mathfrak{g}$-module is equivalent to the data of a $U(\mathfrak{g})$-module.

The algebra $U(\mathfrak{g})$ is a Hopf algebra with coproduct Δ, antipode S and counit ε, given by $\Delta(x) = x \otimes 1 + 1 \otimes x$, $S(x) = -x$, $\varepsilon(x) = 0$, for $x \in \mathfrak{g}$. Let M, N be $\mathfrak{g}$-modules. Let $M \otimes N$ be the vector space tensor product. We make $M \otimes N$ a $\mathfrak{g}$-module via the coproduct. Let $\mathbb{1}$ denote the vector space $\mathbb{C}$ with $\mathfrak{g}$-action given by $g \colon 1 \mapsto 0$, for $g \in \mathfrak{g}$. Then the category of $\mathfrak{g}$-modules is monoidal with unit object $\mathbb{1}$.

Let M and N be $\mathfrak{g}$-modules. Define

$$\mathrm{flip}_{MN} \colon M \otimes N \to N \otimes M, \quad m \otimes n \mapsto n \otimes m.$$

Then flip_{MN} is a $\mathfrak{g}$-module isomorphism. Let V be a $\mathfrak{g}$-module and let

$$V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$$

with $\mathfrak{g}$-action defined by $\langle x \cdot \phi, v \rangle = \langle \phi, S(x) v \rangle$, $x \in U(\mathfrak{g})$, $\phi \in V^*$, $v \in V$. If V is finite dimensional, define

$$\mathrm{ev}_V \colon V^* \otimes V \to \mathbb{1}, \quad \phi \otimes v \mapsto \phi(v), \qquad \mathrm{coev}_V \colon \mathbb{1} \to V \otimes V^*, \quad 1 \mapsto \sum_i v_i \otimes v_i^*,$$

where $\{v_i\}_i$ and $\{v_i^*\}_i$ are dual bases of V and V^* respectively. The maps coev_V and ev_V are $\mathfrak{g}$-module homomorphisms and are precisely the data of an adjunction $(V^* \otimes -, V \otimes -)$. Set $\mathrm{ev}'_V = \mathrm{ev}_V \circ \mathrm{flip}_{VV^*} \colon V \otimes V^* \to \mathbb{1}$ and $\mathrm{coev}'_V = \mathrm{flip}_{VV^*} \circ \mathrm{coev}_V \colon \mathbb{1} \to V^* \otimes V$. Then ev'_V and coev'_V define an adjunction $(V \otimes -, V^* \otimes -)$. That is, the functors $V^* \otimes -$ and $V \otimes -$ are left and right adjoint to each other.

6.3. **Verma modules.** Let $\mathfrak{g}$ be a finite dimensional, simple Lie algebra over $\mathbb{C}$. Let $\lambda \in \mathfrak{h}^*$. Let $\mathbb{C}_\lambda$ be the one dimensional $\mathfrak{b}$-module on which $\mathfrak{n}$ acts trivially and $\mathfrak{h}$ acts via λ. The *Verma module* $M(\lambda)$ is the $\mathfrak{g}$-module representing the functor $\mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda, -)$ from $\mathfrak{g}$-modules to vector spaces. That is, there is a natural isomorphism $\mathrm{Hom}_{\mathfrak{g}}(M(\lambda), -) \simeq \mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda, -)$. By the Yoneda lemma (Lemma 1.2), $M(\lambda)$ is defined uniquely up to canonical isomorphism. Concretely, $M(\lambda) \simeq U(\mathfrak{n}^-) \otimes_{\mathfrak{b}} \mathbb{C}_\lambda$. By construction, $M(\lambda)$ is generated over $\mathfrak{g}$ by a vector v_λ^+

which is annihilated by $\mathfrak{n}$ and on which $\mathfrak{h}$ acts via λ. For details, see [Bou75, Ch. VIII §6 n°3], note that in [Bou75] Verma modules are denoted $Z(\lambda)$.

Let M be a $\mathfrak{g}$-module. The *λ-weight space* of M is $M_\lambda = \{v \in M \mid hv = \lambda(h)v, \text{ for all } h \in \mathfrak{h}\}$. Further, a vector $v \in M_\lambda$ is a *highest weight vector* if $U(\mathfrak{n})$ acts as 0 on v. For instance, the action of $U(\mathfrak{h})$ on $M(\lambda)$ is diagonalizable and the vector $v_\lambda^+ \in M(\lambda)$ is a highest weight vector. As $M(\lambda)_\lambda = \mathbb{C}$, we deduce $\mathrm{End}_{\mathfrak{g}}(M(\lambda)) = \mathbb{C}$.

A *highest weight module* with *highest weight* λ is a module that is isomorphic to a quotient of $M(\lambda)$. Each $M(\lambda)$ admits a unique irreducible quotient, denoted $L(\lambda)$. Further, for $\lambda \neq \mu$, the modules $L(\lambda)$ and $L(\mu)$ are non-isomorphic. See [Bou75, Ch. 8 §6, Thm. 1] for details.

Lemma 6.3. *Let $\mathfrak{g}$ be a simple Lie algebra. Let $\lambda \in \mathfrak{h}^*$, let α be a simple root and let $s_\alpha \in W$ denote the reflection corresponding to α. If $\langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}$, then there is an injection $M(s_\alpha \cdot \lambda) \hookrightarrow M(\lambda)$.*

Proof. This is essentially the consequence of an explicit computation in the special case that $\mathfrak{g} = \mathfrak{sl}_2$ (the Lie algebras of 2×2 traceless matrices), see [Dix, Prop. 7.1.15]. □

6.4. **The Harish-Chandra isomorphism.** Let $\mathfrak{z}$ denote the center of $U(\mathfrak{g})$, let $S(\mathfrak{h})$ denote the algebra of regular functions on $\mathfrak{h}^*$ and let $\lambda \in \mathfrak{h}^*$. Then each $z \in \mathfrak{z}$ defines an element of $\mathrm{End}_{\mathfrak{g}}(M(\lambda))$. As $\mathrm{End}_{\mathfrak{g}}(M(\lambda)) = \mathbb{C}$, allowing λ to vary, we obtain an algebra homomorphism from $\mathfrak{z}$ to complex valued functions on $\mathfrak{h}^*$. It is not too involved to show (for instance, see [Bou75, Ch, VIII §6, Prop. 7])) that this is in fact a homomorphism from $\mathfrak{z}$ to polynomial functions on $\mathfrak{h}^*$. Thus, we have an algebra homomorphism $\chi\colon \mathfrak{z} \to S(\mathfrak{h})$, where $S(\mathfrak{h})$ is the symmetric algebra of the vector space $\mathfrak{h}$. Let $\lambda \in P^+ - \rho$ and write χ_z for the image of z under χ. Then Lemma 6.3 implies that $\chi_z(\lambda) = \chi_z(s \cdot \lambda)$, for each simple reflection $s \in W$. Thus, χ_z is constant on the W-dot orbits of $P^+ - \rho$. These orbits are Zariski dense in $\mathfrak{h}^*$. Thus, the image of χ lies in the W-invariants (dot action) of $S(\mathfrak{h})$. In fact:

Theorem 6.4 (Harish-Chandra isomorphism)**.** *The map $\chi\colon \mathfrak{z} \to S(\mathfrak{h})^{W\cdot}$ is an algebra isomorphism.*

Proof. See [Bou75, Ch. VIII, §8, Thm. 2]. □

Proposition 6.5. *Let $\lambda, \mu \in \mathfrak{h}^*$. Suppose λ and μ lie in distinct orbits under the dot action of W. Then $\mathrm{Ext}^1_{\mathfrak{g}}(L(\lambda), L(\mu)) = 0$.*

Proof. As $L(\lambda)$ and $L(\mu)$ are quotients of $M(\lambda)$ and $M(\mu)$ respectively, it follows that each element in $\mathfrak{z}$ acts as scalar multiplication on both $L(\lambda)$ and $L(\mu)$. The Harish-Chandra isomorphism implies that there exists $z \in \mathfrak{z}$ such that z acts on $L(\lambda)$ and $L(\mu)$ via multiplication by distinct scalars. Consequently, any exact sequence $0 \to L(\mu) \to M \to L(\lambda) \to 0$ must split. □

Proposition 6.6. *Each Verma module has a finite composition series.*

Proof. As each W-orbit in $\mathfrak{h}^*$ contains only finitely many elements, Proposition 6.5 implies that there are only finitely many distinct $\mu \in \mathfrak{h}^*$ such that $L(\mu)$ is a composition factor of $M(\lambda)$. So it suffices to show that $L(\mu)$ occurs as a composition factor of $M(\lambda)$ with finite multiplicity. This is a consequence of the fact that each weight space of a Verma module is finite dimensional. See [Dix, Prop. 7.6.1 (i)] for more details. □

6.5. **Morphisms between Verma modules.**

Proposition 6.7. *Let $\lambda, \mu \in \mathfrak{h}^*$, then*

(i) *the Verma module $M(\lambda)$ contains a unique simple submodule;*
(ii) *any homomorphism $M(\lambda) \to M(\mu)$ is either zero or injective;*
(iii) $\dim(\mathrm{Hom}_{\mathfrak{g}}(M(\lambda), M(\mu)) \leq 1$*;*

Proof. For (i) see [Dix, Prop. 7.6.3]. For (ii) and (iii) see [Dix, Thm. 7.6.6]. □

Corollary 6.8. *Let $\lambda \in P^+ - \rho$ and let $w \in W$. Then $M(w \cdot \lambda) \hookrightarrow M(\lambda)$.*

Proof. Let $w = s_1 \cdots s_m$ be a reduced expression for w and write α_i for the simple root corresponding to s_i. In view of Lemma 6.3, it suffices to show that $\langle (s_i \cdots s_m \cdot \lambda) + \rho, \alpha_{i-1}^\vee \rangle \in \mathbb{Z}_{\geq 0}$, for each $1 < i \leq m$. Now $\langle (s_i \cdots s_m \cdot \lambda) + \rho, \alpha_{i-1}^\vee \rangle = \langle \lambda + \rho, s_m \cdots s_i(\alpha_{i-1}^\vee) \rangle$. Proposition 6.1 (ii) implies that $s_m \cdots s_i(\alpha_{i-1}^\vee) = \alpha^\vee$ for some $\alpha \in R^+$. This gives the result. □

Proposition 6.9. *Let $\lambda \in P^+ - \rho$, let w_0 denote the longest element in the Weyl group and let $w \in W$ be arbitrary. Then $M(w_0 \cdot \lambda)$ is the unique simple submodule of $M(w \cdot \lambda)$.*

Proof. Corollary 6.8 implies that $M(w_0 \cdot \lambda) \hookrightarrow M(w \cdot \lambda)$. Furthermore, since all weights of $M(w_0 \cdot \lambda)$ are lower than $w_0 \cdot \lambda$, Proposition 6.5 implies that $M(w_0 \cdot \lambda)$ is simple. □

6.6. **Category $\mathcal{O}$.** The Bernstein-Gelfand-Gelfand category $\mathcal{O}$ is the full subcategory of $U(\mathfrak{g})$-modules M such that

- the action of $U(\mathfrak{b})$ on M is locally finite, i.e., for any $v \in M$,
$$\dim(U(\mathfrak{n})v) < \infty;$$
- M is semisimple as a $U(\mathfrak{h})$-module;
- M is finitely generated over $U(\mathfrak{g})$.

Proposition 6.10 ([BGG])**.** *Every object in $\mathcal{O}$ has finite length.*

Proof. The axioms for $\mathcal{O}$ imply that the action of $\mathfrak{n}$ on a module in $\mathcal{O}$ is locally nilpotent. That is, for any vector v in the given module, there exists $n \in \mathbb{Z}_{\geq 0}$ (dependant on v) such that $(x_1 \cdots x_n)v = 0$, for any $x_i \in \mathfrak{n}$. This in turn implies that every object of $\mathcal{O}$ admits a finite filtration whose factors are highest weight modules. Proposition 6.6 implies that highest weight modules have finite length. This gives the result. □

Proposition 6.11 ([BGG])**.** *Let $L \in \mathcal{O}$ be a simple object. Then $L \simeq L(\lambda)$, for some $\lambda \in \mathfrak{h}^*$.*

Proof. A simple object in $\mathcal{O}$ must be a highest weight module. As a Verma module admits a unique simple quotient, the result follows. For details, see [Bou75, Ch. VIII §6, Prop. 5]. □

Proposition 6.12. *Let $\lambda, \mu \in \mathfrak{h}^*$ be such that $\lambda \not< \mu$. Let $M \in \mathcal{O}$ be a highest weight module with highest weight μ. Then $\mathrm{Ext}^1_{\mathcal{O}}(M(\lambda), M) = 0$.*

Proof. Suppose $0 \to M \to M' \xrightarrow{f} M(\lambda) \to 0$ is an exact sequence. Let $v' \in M'_\lambda$ be such that $f(v') = v_\lambda^+$. As all weights ν of M satisfy $\nu \leq \mu$, we deduce that v' is a highest weight vector in M'. Let $M'' \subseteq M'$ be the $U(\mathfrak{g})$-module generated by v', in particular M' is isomorphic to a quotient of $M(\lambda)$. The morphism f maps M'' onto $M(\lambda)$, consequently $M(\lambda)$ is a quotient of M''. Thus, $f\colon M'' \to M(\lambda)$ is an isomorphism and the above short exact sequence splits. $\square$

6.7. **Duality.** Let $\tau\colon \mathfrak{g} \to \mathfrak{g}$ be the Cartan involution on $\mathfrak{g}$. This is the unique automorphism of $\mathfrak{g}$ that acts as -1 on $\mathfrak{h}$ and maps $\mathfrak{g}_\alpha \to \mathfrak{g}_{-\alpha}$, for all $\alpha \in R^+$. For example, the Cartan involution on $\mathfrak{sl}_n$, the Lie algebra of $n \times n$ traceless matrices, is given by sending a matrix to the negative of its transpose. See [Ja, §1.2] for details. Note that τ is '$-\sigma$' in the notation of [Ja].

To $M \in \mathcal{O}$ we associate a module $M^\vee = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda^*$, with $\mathfrak{g}$-module structure given by $\langle x \cdot \phi, v\rangle = \langle \phi, \tau(S(x))v\rangle$, $x \in \mathfrak{g}, \phi \in M^\vee, v \in M$. Suppose $f \in \mathrm{Hom}_{\mathcal{O}}(M,N)$, define $f^\vee\colon N^\vee \to M^\vee$ via $\langle f^\vee(\psi), v\rangle = \langle \psi, f(v)\rangle$, $\psi \in N^\vee, v \in M$. This defines a contravariant functor ${}^\vee\colon \mathcal{O} \to \mathcal{O}$. The involution τ and the antipode S clearly commute with each other. Further, since S is also an involution, it follows that ${}^{\vee\vee} \simeq \mathrm{id}$.

Proposition 6.13. *Let $L \in \mathcal{O}$ be simple. Then $L^\vee \simeq L$.*

Proof. As ${}^\vee$ is an equivalence, it must send a simple module to a simple module. Further, $L^\vee$ has the same weight spaces as L, since $\tau \circ S$ fixes $\mathfrak{h}$ pointwise. In particular, the highest weight vectors in $L^\vee$ and L have the same weight. Thus, $L^\vee \simeq L$, since simple modules in $\mathcal{O}$ are determined by their highest weight (see Prop. 6.11). $\square$

Extend ${}^\vee$ to a contravariant functor $\mathbb{D}$ on $\mathrm{Comp}(\mathcal{O})$ as follows. Given a complex $X^\bullet = \cdots \to X^i \xrightarrow{d_i} X^{i+1} \to \cdots$ in $\mathcal{O}$ with X^i concentrated in degree i, let $\mathbb{D}X^\bullet$ to be the complex $\cdots \to (X^{i+1})^\vee \xrightarrow{d_i^\vee} (X^i)^\vee \to \cdots$ with X^i concentrated in degree $-i$. By definition, $\mathbb{D} \circ [1] = [-1] \circ \mathbb{D}$.

As ${}^\vee$ is a contravariant equivalence, the module $M(\lambda)^\vee$ has a unique simple submodule that is isomorphic to $L(\lambda)$.

Proposition 6.14. *Let $\lambda \in \mathfrak{h}^*$. Then* $\mathrm{Hom}_{\mathfrak{g}}(M(\lambda), M(\mu)^\vee) = \begin{cases} \mathbb{C} & \text{if } \lambda = \mu; \\ 0 & \text{otherwise.} \end{cases}$

Proof. Suppose $\mathrm{Hom}_{\mathfrak{g}}(M(\lambda), M(\mu)^\vee) \neq 0$. Let v^+ be a highest weight vector in $M(\mu)^\vee$, then $\mathrm{wt}(v^+) \leq \mu$. Consequently, $\lambda \leq \mu$. As $\mathrm{Hom}_{\mathfrak{g}}(M(\lambda), M(\mu)^\vee) = \mathrm{Hom}_{\mathfrak{g}}(M(\mu), M(\lambda)^\vee)$, the same argument implies that $\mu \leq \lambda$. Hence, $\lambda = \mu$. On the other hand, $M(\lambda)^\vee$ contains a unique (up to scalars) highest weight vector of weight λ. Thus, $\mathrm{Hom}_{\mathfrak{g}}(M(\lambda), M(\lambda)^\vee) = \mathbb{C}$. $\square$

Lemma 6.15. *Let $\lambda, \mu \in \mathfrak{h}^*$, then $\mathrm{Ext}^1_{\mathcal{O}}(M(\lambda), M(\mu)^\vee) = 0$.*

Proof. Let $0 \to M(\mu)^\vee \to M \to M(\lambda) \to 0$ be a short exact sequence. If $\lambda \not< \mu$, then this sequence splits by Proposition 6.12. If $\lambda < \mu$, then applying ${}^\vee$ and using the same argument as above gives that the sequence splits. $\square$

6.8. **Verma and dual Verma filtrations.** A *Verma filtration* of a $\mathfrak{g}$-module M is a filtration $0 \subset V_1 \subset \cdots \subset V_m = M$ such that each V_{i+1}/V_i is isomorphic to a Verma module. The filtration is a *dual Verma filtration* if each V_{i+1}/V_i is isomorphic to the ${}^\vee$-dual of a Verma module.

Proposition 6.16. *Let $M \in \mathcal{O}$ and assume that M has a Verma filtration.*

(i) *If λ is maximal among the weights of M, then M has a submodule isomorphic to $M(\lambda)$ and $M/M(\lambda)$ has a Verma filtration.*
(ii) *If $M \simeq M' \oplus M''$, then both M' and M'' have Verma filtrations.*

Proof.

(i) The assumption on λ implies the existence of a non-zero morphism $\phi\colon M(\lambda) \to M$. We claim that this map is injective. Let $0 \subset V_1 \subset \cdots \subset V_m = M$ be the given Verma filtration and let i be minimal with the property that $\phi(M(\lambda)) \subseteq V_i$ and let $V_i/V_{i-1} \simeq M(\mu)$. Then we have an induced map $\phi'\colon M(\lambda) \to V_i/V_{i-1} \simeq M(\mu)$. So $\lambda \leq \mu$; as λ was chosen to be maximal, this implies $\lambda = \mu$. We deduce that ϕ' is an isomorphism. Thus, ϕ is injective and we may write $M(\lambda) \subseteq V$. As $M(\lambda) \cap V_{i-1} = \ker \phi' = 0$, we obtain an exact sequence $0 \to V_{i-1} \to V/M(\lambda) \to V/V_i \to 0$. Both V_{i-1} and V/V_i have Verma filtrations, which combine to produce a Verma filtration for $V/M(\lambda)$.
(ii) Proceed by induction on the length of the Verma filtration of M. If this length is 1, i.e., M is a Verma module, then as Verma modules are indecomposable, there is nothing to prove. Now assume that the length of the Verma filtration is greater than 1. Let λ be maximal among the weights of M. We may assume that $M'_\lambda \neq 0$. Arguing as in (i), there is an injective map $M(\lambda) \hookrightarrow V'$. Further, $V/M(\lambda) \simeq V'/M(\lambda) \oplus V''$. Now applying the induction hypothesis gives the result. □

Lemma 6.17. *Let $\lambda \in \mathfrak{h}^*$. Assume $M \in \mathcal{O}$ has a Verma filtration. Then* $\operatorname{Ext}^1_{\mathcal{O}}(M, M(\lambda)^\vee) = 0$.

Proof. The hypothesis implies that M has filtration by Verma modules in the sense of triangulated categories (see Section 3 §3.1). Combining this with Lemma 6.15 gives the result. □

6.9. **Tensoring with finite dimensional modules.** Category $\mathcal{O}$ is not necessarily closed under arbitrary tensor products. However, tensoring with a finite dimensional module does preserve $\mathcal{O}$. More precisely:

Proposition 6.18 (Tensor identity)**.** *Let M be a $\mathfrak{b}$-module and let V be a finite dimensional $\mathfrak{g}$-module. Then there is a natural isomorphism*

$$V \otimes (U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} M) \simeq U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (V \otimes M),$$

where the left hand side is a tensor product of $\mathfrak{g}$-modules and the right hand side is the $\mathfrak{g}$-module induced from the $\mathfrak{b}$-module $V \otimes M$.

Proof. As induction is left adjoint to restriction and $V \otimes -$ is left adjoint to $V^* \otimes -$, we have natural isomorphisms

$$\begin{aligned}
\operatorname{Hom}_{\mathfrak{g}}(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (V \otimes M), -) &\simeq \operatorname{Hom}_{\mathfrak{b}}(V \otimes M, -) \\
&\simeq \operatorname{Hom}_{\mathfrak{b}}(M, V^* \otimes -) \\
&\simeq \operatorname{Hom}_{\mathfrak{g}}(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} M, V^* \otimes -) \\
&\simeq \operatorname{Hom}_{\mathfrak{g}}(V \otimes (U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} M), -).
\end{aligned}$$

This implies the result by the Yoneda lemma (Lemma 1.2). □

Corollary 6.19. *Let V be a finite dimensional $\mathfrak{g}$-module and let $\lambda \in \mathfrak{h}^*$. Then the module $V \otimes M(\lambda)$ admits a filtration $0 = V_0 \subset V_1 \subset \cdots \subset V_m = V \otimes M(\lambda)$ which satisfies the following:*

(i) *each subquotient V_i / V_{i-1} is isomorphic to a Verma module;*
(ii) *the Verma module $M(\lambda + \mu)$ occurs with multiplicity $\dim(V_\mu)$ as a quotient of this filtration;*
(iii) *if $V_i / V_{i-1} \simeq M(\nu)$ and $V_j / V_{j-1} \simeq M(\nu')$ with $\nu \geq \nu'$, then $i < j$.*

Proof. The tensor identity gives that $V \otimes M(\lambda) \simeq U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (V \otimes \mathbb{C}_\lambda)$. Let $\{v_1, \ldots, v_m\}$ be a weight basis of V ordered so that $\mathrm{wt}(v_i) \geq \mathrm{wt}(v_j)$ implies that $i <\leq j$. Let $N_i = \{\sum_{j=1}^{i} v_i \otimes \mathbb{C}_\lambda\}$. This defines a filtration of $U(\mathfrak{b})$-modules $0 = N_0 \subset N_1 \subset \cdots \subset N_m = V \otimes \mathbb{C}_\lambda$ with 1-dimensional quotients. Finally, observe that the functor $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} -$ from $\mathfrak{b}$-modules to $\mathfrak{g}$-modules is exact, since $U(\mathfrak{g})$ is free over $U(\mathfrak{b})$. It follows that the induced filtration $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} N_i$ on $V \otimes M$ has the required properties. □

6.10. **Blocks.** The action of $\mathfrak{z}$, the center of $U(\mathfrak{g})$, decomposes $\mathcal{O}$ into blocks, i.e., $\mathcal{O} = \bigoplus_\chi \mathcal{O}_\chi$, where χ runs through central characters. As each Verma module is indecomposable, it belongs to some $\mathcal{O}_\chi$. Via the Harish-Chandra isomorphism, we index the blocks of $\mathcal{O}$ by W-orbits (dot action) in $\mathfrak{h}^*$. The Verma modules $M(\lambda)$ and $M(\mu)$ belong to the same block if and only if $\lambda \in W \cdot \mu$. See [BGG, §3] for details.

Write $\mathcal{O}_\lambda$ for the block containing $M(\lambda)$. The block $\mathcal{O}_0$ is the *principal block*. Since the duality functor $^\vee$ preserves simple modules, blocks are also preserved by $^\vee$.

6.11. **Projectives and injectives.** Let $M \in \mathcal{O}$ and suppose $P_M \twoheadrightarrow M$ is the projective cover of M. Then we obtain an injection $M^\vee \hookrightarrow P_M^\vee$. As $^\vee$ is an equivalence, it follows that $P_M^\vee$ is the injective hull of $M^\vee$. Thus, to show that $\mathcal{O}$ contains enough projectives and injectives, it is sufficient to show that $\mathcal{O}$ contains enough projectives. Since each object in $\mathcal{O}$ has finite length, this is further reduced to showing that each simple object in $\mathcal{O}$ admits a projective cover. It is easy to get started:

Proposition 6.20. *Let $\lambda \in \mathfrak{h}^*$ be such that $\langle \lambda + \rho, \alpha^\vee \rangle \geq 0$ for each $\alpha \in R^+$. Then $M(\lambda)$ is projective in $\mathcal{O}$.*

Proof. It suffices to show that $M(\lambda)$ is projective in $\mathcal{O}_\lambda$. By hypothesis, λ is maximal amongst weights occuring in its W-dot orbit. The result now follows from Proposition 6.12. □

Proposition 6.21. *Category $\mathcal{O}$ has enough projectives. Further, if $P \in \mathcal{O}$ is projective then P admits a Verma filtration.*

Proof. See [BGG, Thm. 1] and [BGS, §3.2]. □

Write $P(\lambda)$ (resp. $I(\lambda)$) for the projective cover (resp. injective hull) of $L(\lambda)$.

Proposition 6.22. *Each object in $\mathcal{O}$ admits a bounded projective resolution.*

Proof. This is [BGG, Thm. 6]. □

6.12. **Homological criteria for Verma and dual Verma filtrations.**

Proposition 6.23. *Let $\lambda \in \mathfrak{h}^*$ and let $i \neq 0$. Assume $M \in \mathcal{O}$ has a Verma filtration. Then $\mathrm{Ext}^i_{\mathcal{O}}(M, M(\lambda)^\vee) = 0$.*

Proof. There is nothing to prove if $i < 0$. For $i > 0$, proceed by induction on i. The case $i = 1$ is provided by Lemma 6.17. Let's show that $\mathrm{Ext}^{i+1}_{\mathcal{O}}(M, M(\lambda)^\vee) = 0$. Here we proceed by induction on the filtration length of M. In the base case $M = M(\mu)$, for some $\mu \in \mathfrak{h}^*$, and we have a short exact sequence $0 \to N \to P(\mu) \to M \to 0$. Prop. 6.21 implies that $P(\mu)$ and N have Verma filtrations. This gives an exact sequence $\mathrm{Ext}^i_{\mathcal{O}}(N, M(\lambda)^\vee) \to \mathrm{Ext}^{i+1}_{\mathcal{O}}(M, M(\lambda)^\vee) \to \mathrm{Ext}^{i+1}_{\mathcal{O}}(P(\mu), M(\lambda)^\vee)$. The first term is 0 by the induction hypothesis on i and the last term is 0 since $P(\mu)$ is projective. This forces the middle term to vanish.

For arbitrary M, the same argument as above applies, with $P(\lambda)$ replaced with a projective cover of M. Alternatively, proceed by induction on the length of the Verma filtration of M. □

Proposition 6.24. *Let $M \in \mathcal{O}$. The following are equivalent.*

(i) *M has a Verma filtration.*
(ii) *$\mathrm{Ext}^i_{\mathcal{O}}(M, M(\lambda)^\vee) = 0$ for all $i \neq 0$ and all $\lambda \in \mathfrak{h}^*$.*
(iii) *$\mathrm{Ext}^1_{\mathcal{O}}(M, M(\lambda)^\vee) = 0$ for all $\lambda \in \mathfrak{h}^*$.*

Proof. That (i) implies (ii) is the previous proposition. That (ii) implies (iii) is obvious. That (iii) implies (i) is [Donkin, Prop. A.2.2 (iii)]. □

Proposition 6.25 ([BGG, Thm. 5], BGG reciprocity)**.** *Let $\lambda, \mu \in \mathfrak{h}^*$. Then $[M(\lambda) : L(\mu)] = [P(\mu) : M(\lambda)^\vee]$.*

Proof. Define a bilinear form

$$\langle \cdot, \cdot \rangle \colon K_0(\mathcal{O}) \otimes K_0(\mathcal{O}) \to \mathbb{Z}, \quad \langle [M], [N] \rangle = \sum_i (-1)^i \dim(\mathrm{Ext}^i_{\mathcal{O}}(M, N)).$$

This form is symmetric, since $\langle [M], [N] \rangle = \langle [\mathbb{D}N], [\mathbb{D}M] \rangle = \langle [N], [M] \rangle$ for each $M, N \in \mathcal{O}$. By Prop. 6.23, $\langle [M(\lambda)], [M(\mu)^\vee] \rangle = \delta_{\lambda,\mu}$. Further, $\langle [L(\lambda)], [I(\mu)] \rangle = \delta_{\lambda,\mu}$. So $[M(\lambda) : L(\mu)] = \langle [M(\lambda)], [I(\mu)] \rangle = \langle [P(\mu)], [M(\lambda)^\vee] \rangle = [P(\mu) : M(\lambda)^\vee]$. □

6.13. **Tilting modules.** An object $T \in \mathcal{O}$ is *tilting* if T admits a Verma as well as a dual Verma filtration. Proposition 6.16 (ii) implies that direct summands of tilting objects are again tilting. As every object in $\mathcal{O}$ has finite length, it follows that each tilting module in $\mathcal{O}$ can expressed as a direct sum of indecomposable tilting modules.

Proposition 6.26. *For each $\lambda \in \mathfrak{h}^*$ there is a unique (up to isomorphism) indecomposable tilting object $T(\lambda)$ such that $T(\lambda)$ admits a Verma filtration, starting at $M(\lambda)$.*

Proof. See [So98, Thm. 5.2]. □

Corollary 6.27. *Let T be an indecomposable tilting object in $\mathcal{O}$. Then $T \simeq T(\lambda)$ for some $\lambda \in \mathfrak{h}^*$.*

Proof. As T is tilting, it admits a Verma filtration starting at say $M(\lambda)$. Then the uniqueness of $T(\lambda)$ implies that $T \simeq T(\lambda)$. □

7. Translation functors

7.1. **Translation.** Let $\lambda, \mu \in P$. Let ν be the unique element in $W(\mu - \lambda) \cap P^+$. Let $\mathrm{pr}_\lambda \colon \mathcal{O} \to \mathcal{O}$ denote the functor of projection onto the $\mathcal{O}_\lambda$ block. In particular, pr_λ is left and right adjoint to itself. The *translation functor* $T_\lambda^\mu \colon \mathcal{O} \to \mathcal{O}$ is defined as the composition $T_\lambda^\mu = \mathrm{pr}_\mu \circ (L(\nu) \otimes -) \circ \mathrm{pr}_\lambda$. Note that T_λ^μ may also be interpreted as a functor $\mathcal{O}_\lambda \to \mathcal{O}_\mu$.

Let w_0 be the longest element in W, then $w_0\nu$ is the unique element in $W(\lambda - \mu) \cap P^+$. Since ν is in P^+, $L(\nu)$ is finite dimensional. Further. $L(-w_0\nu) \simeq L(\nu)^*$ [Bou75, Ch. VIII §7, Prop. 11 (ii)], we infer that T_μ^λ is left and right adjoint to T_λ^μ (see Ch. 6 §6.2).

Proposition 7.1. *Let $w \in W$. Let $\lambda, \mu \in P^+$. Let W_λ (resp. W_μ) denote the stabilizer of λ (resp. μ) under the dot action.*

(i) *If $W_\mu \subseteq W_\lambda$, then $T_\mu^\lambda M(w \cdot \mu) = M(w \cdot \lambda)$.*

(ii) *If $W_\mu \subseteq W_\lambda$, then* $T_\mu^\lambda L(w \cdot \mu) = \begin{cases} L(w \cdot \lambda) & \text{if } w \cdot \mu \le w \cdot \lambda; \\ 0 & \text{otherwise.} \end{cases}$

(iii) *If $W_\lambda \subseteq W_\mu$, then $T_\mu^\lambda M(w \cdot \mu)$ has a Verma filtration $0 \subset V_1 \subset \cdots \subset V_m = T_\mu^\lambda$ such that: if $V_{i+1}/V_i \simeq M(\nu)$, $V_{j+1}/V_j \simeq M(\nu')$ and $\nu \ge \nu'$, then $i \le j$. Furthermore, the multiplicities in this filtration are given by*

$$[T_\mu^\lambda M(w \cdot \mu) : M(ww' \cdot \lambda)] = \begin{cases} 1 & \text{if } w' \text{ is a coset representative of } W_\mu / W_\lambda; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. (i) is [Ja, Satz 2.10 (i)], (ii) is [Ja, Thm. 2.11] and (iii) is [Ja, Satz 2.17] (also see Prop. 6.18). □

Corollary 7.2 (Translation principle)**.** *Let $\lambda, \mu \in P^+ - \rho$. Suppose λ and μ have the same stabilizer relative to the dot action of W. Then $T_\lambda^\mu \colon \mathcal{O}_\lambda \to \mathcal{O}_\mu$ and $T_\mu^\lambda \colon \mathcal{O}_\mu \to \mathcal{O}_\lambda$ are mutually inverse equivalences.*

Proof. Since the classes of Verma modules in $K_0(\mathcal{O}_\lambda)$ and $K_0(\mathcal{O}_\mu)$ constitute bases for their respective Grothendieck groups, the result follows from Prop. 7.1 (i) and Prop. 5.2. □

7.2. **Wall crossing.** Let $s \in W$ be a simple reflection. Fix $\nu \in P^+ - \rho$ such that the stabilizer of ν under the dot action is $\{e, s\}$. Set $\pi_{s*} = T_0^\nu$ and $\pi_s^* = T_\nu^0$. Fix adjunctions (π_s^*, π_{s*}) and (π_{s*}, π_s^*). Write ε_s for the counit of the pair (π_s^*, π_{s*}) and η_s' for the unit of the pair (π_{s*}, π_s^*).

Let $w \in W$ and let w_0 be the longest element in W_0. We adopt the following notational convention:

$$M(w) = M(w_0 w^{-1} \cdot 0).$$

Further, let $M(w)^\vee, L(w), P(w), I(w), T(w)$ have the obvious meanings. In particular, $M(e) \simeq L(e) \simeq M(e)^\vee$.

Set

$$\Theta_s^* = 0 \to \pi_s^* \pi_{s*} \xrightarrow{\varepsilon} \mathrm{id} \to 0 \quad \text{and} \quad \Theta_s^! = 0 \to \mathrm{id} \xrightarrow{\eta'} \pi_s^* \pi_{s*} \to 0,$$

with $\pi_s^* \pi_{s*}$ in degree 0 in both cases.

Remark 7.3. In the above form, the definition of Θ_s^* and $\Theta_s^!$ is due to R. Rouquier (see [Ro, §4.1.5]). However, also see [Ri] and Ch. 8 §8.2 and §8.3.

Proposition 7.4. *The functors Θ_s^* and $\Theta_s^!$ are mutually inverse self-equivalences of $\mathrm{D}^{\mathrm{b}}(\mathcal{O}_0)$.*

Proof. Prop. 7.1 (i) and (iii) imply that at the level of $K_0(\mathcal{O}_0)$, $[\pi_s^*\pi_{s*}M(w)] = [M(w)] + [M(sw)]$. As the classes of Verma modules give a basis of $K_0(\mathcal{O}_0)$, we deduce that $[\pi_s^*\pi_{s*}\pi_s^*\pi_{s*}X] = 2[\pi_s^*\pi_{s*}X]$ for all $X \in \mathcal{O}_0$. Applying Thm. 5.4 gives the desired result. □

Remark 7.5. The functor Θ_s^* is precisely the cocone of the morphism $\varepsilon\colon \pi_s^*\pi_{s*} \to \mathrm{id}$ and the functor $\Theta_s^!$ is precisely the cone of the morphism $\eta'\colon \mathrm{id} \to \pi_s^*\pi_{s*}$ (see (1.2) and Ch. 3). Further, via the identification of $K_0(\mathcal{O}_0)$ with $K_0(\mathrm{D}^{\mathrm{b}}(\mathcal{O}_0))$ (see Ch. 3 §3.4), we have that at the level of $K_0(\mathcal{O}_0)$:

$$[\Theta_s^*X] = [\pi_s^*\pi_{s*}X] - [X] = [\Theta_s^!X] \quad \text{for all } X \in \mathcal{O}.$$

Lemma 7.6. *Let $s \in W$ be a simple reflection and let $w \in W$ be arbitrary.*

(i) *The morphism $\eta'\colon M(w) \to \pi_s^*\pi_{s*}M(w)$ is injective.*
(ii) *The morphism $\varepsilon\colon \pi_s^*\pi_{s*}M(w)^\vee \to M(w)^\vee$ is surjective.*

Proof. We will only show (i), the proof of (ii) is similar. Prop. 7.1 (i) and (iii) imply that $\pi_s^*\pi_{s*}M(w)$ is non-zero and has a Verma filtration. From Prop. 6.7 (iii) we infer that $\eta_s'\colon M(w) \to \pi_s^*\pi_{s*}M(w)$ is either zero or injective. Lemma 2.1 implies that the map is injective. □

Proposition 7.7. *Let $s \in W$ be a simple reflection and let $M \in \mathcal{O}_0$.*

(i) *If M admits a Verma filtration, then $\Theta_s^!M$ is in $\mathcal{O}_0$, i.e., the complex $\Theta_s^!M$ has cohomology concentrated in degree 0.*
(ii) *If M admits a dual Verma filtration, then Θ_s^*M is in $\mathcal{O}_0$.*

Proof. Follows from Lemma 7.6. □

Proposition 7.8. *Let $s \in W$ be a simple reflection and let $w \in W$.*

(i) *If $w < sw$, then $\Theta_s^*M(w) \simeq M(sw)$.*
(ii) *If $w < sw$, then $\Theta_s^!M(w)^\vee \simeq M(sw)^\vee$.*
(iii) *If $sw < w$, then $\Theta_s^!L(w) \simeq L(w)[1]$ (or equivalently $\Theta_s^*L(w) \simeq L(w)[-1]$).*

Proof. If $w < sw$, then Prop. 7.1 (iii) implies that $\pi_s^*\pi_{s*}M(sw)$ represents a class in $\mathrm{Ext}^1_{\mathcal{O}_0}(M(w), M(sw))$. Combining this with Lemma 7.6 (i) we deduce that $\Theta_s^!M(sw) \simeq M(w)$. This gives (i). The proof of (ii) is similar. For (iii), we observe that if $sw < w$, then Prop. 7.1 (ii) implies $\pi_s^*\pi_{s*}L(w) = 0$. So $\Theta_s^!L(w) \simeq L(w)[1]$. □

Remark 7.9. In fact $\mathbb{D} \circ \Theta_s^* = \Theta_s^! \circ \mathbb{D}$. However, we will neither prove nor use this fact.

7.3. First applications: some homological properties of $\mathcal{O}_0$.

Theorem 7.10. *Let $s \in W$ be a simple reflection and let $w, w' \in W$.*

(i) [GJ, §5.2.1] *If $w' < sw'$ and $w < sw$, then $\mathrm{Ext}^i_{\mathcal{O}_0}(M(w'), M(w)) = \mathrm{Ext}^i_{\mathcal{O}_0}(M(sw'), M(sw))$.*
(ii) [Sc, Lemma 5.17] *If $\mathrm{Ext}^i_{\mathcal{O}_0}(M(w'), M(w)) \neq 0$, then $w' \leq w$ and $0 \leq i \leq \ell(w) - \ell(w')$.*

(iii) [Ca, Thm. 3.18] *If* $w' \leq w$*, then* $\mathrm{Ext}^{\ell(w)-\ell(w')}_{\mathcal{O}_0}(M(w'), M(w)) = \mathbb{C}$.

(iv) *If* $w' < sw'$ *and* $sw < w$*, then* $\mathrm{Ext}^{i+1}_{\mathcal{O}_0}(M(w'), L(w)) = \mathrm{Ext}^i_{\mathcal{O}_0}(M(sw'), L(w))$.

(v) *(Bott's Theorem,* [Bott, Thm. 15]*)* $\mathrm{Ext}^i_{\mathcal{O}_0}(M(w), L(w_0)) = \begin{cases} \mathbb{C} & \text{if } i = \ell(ww_0), \\ 0 & \text{otherwise.} \end{cases}$

Proof. (i) is immediate from Prop. 7.4 and Prop. 7.8 (i). For (ii), proceed by downwards induction on $\ell(w')$. If $\ell(w') = \ell(w_0)$, then $w' = w_0$ and $M(w') = P(w')$ (Prop. 6.20). In this case the result is clear. Now let w' be arbitrary and assume the result holds for all w'' such that $\ell(w'') > \ell(w')$. That is, if $\ell(w'') > \ell(w')$, then $\mathrm{Ext}^i_{\mathcal{O}_0}(M(w''), M(w)) \neq 0$ implies $w'' \leq w$ and $0 \leq i \leq \ell(w) - \ell(w'')$. Let s be a simple reflection such that $\ell(sw') > \ell(w')$. In particular, $sw' > w'$ and

$$\mathrm{Ext}^i_{\mathcal{O}_0}(M(w'), M(w)) = \mathrm{Ext}^i_{\mathcal{O}_0}(\Theta^!_s M(sw'), M(w)) = \mathrm{Ext}^i_{\mathcal{O}_0}(M(sw'), \Theta^*_s M(w))$$

by Prop. 7.4 and Prop. 7.8 (i). If $\ell(sw) > \ell(w)$, then $\mathrm{Ext}^i_{\mathcal{O}_0}(M(sw'), \Theta^*_s M(w)) = \mathrm{Ext}^i_{\mathcal{O}_0}(M(sw'), M(sw))$ by Prop. 7.8 (i). By the induction hypothesis, if

$$\mathrm{Ext}^i_{\mathcal{O}_0}(M(sw'), M(sw)) \neq 0,$$

then $sw' \leq sw$ and $0 \leq i\ell(sw) - \ell(sw')$. Since $sw' > w'$ and $sw > w$, this gives $w' \leq w$ and $0 \leq i \leq \ell(w) - \ell(w')$. Thus, to finish the proof it remains to consider the case $\ell(sw) < \ell(w)$. The distinguished triangle $\Theta^*_s M(w) \to \pi^*_s \pi_{s*} M(w) \to M(w) \rightsquigarrow$ gives an exact sequence

$$\mathrm{Ext}^{i-1}_{\mathcal{O}_0}(M(sw'), M(w)) \to \mathrm{Ext}^i_{\mathcal{O}_0}(M(sw'), \Theta^*_s M(w)) \to \mathrm{Ext}^i_{\mathcal{O}_0}(M(sw'), \pi^*_s \pi_{s*} M(w)).$$

If the middle term is non-zero, then the terms on either extreme cannot both be zero. If $\mathrm{Ext}^{i-1}_{\mathcal{O}_0}(M(sw'), M(w)) \neq 0$, then the induction hypothesis implies that $\ell(sw') \leq \ell(w)$ and $0 \leq i - 1 \leq \ell(w) - \ell(sw')$. As $\ell(sw') > \ell(w')$, this gives $\ell(w') \leq \ell(w)$ and $0 \leq i \leq \ell(w) - \ell(w')$. Now consider the case

$$\mathrm{Ext}^i_{\mathcal{O}_0}(M(sw'), \pi^*_s \pi_{s*} M(w)) \neq 0.$$

By Prop. 7.1 (iii) $\pi^*_s \pi_{s*} M(w)$ is filtered (in the sense of triangulated categories, see Ch. 3 §3.1) by $M(w)$ and $M(sw)$. So we must have either $\mathrm{Ext}^i_{\mathcal{O}_0}(M(sw'), M(w)) \neq 0$ or $\mathrm{Ext}^i_{\mathcal{O}_0}(M(sw'), M(sw)) \neq 0$. In both cases the induction hypothesis gives the desired result.

The proof of (iii) is essentially the same as that of (ii). (iv) is immediate from Prop. 7.4, Prop. 7.8 (i) and Prop. 7.8 (iii).

For (vi), let $s_1, \ldots, s_m$ be a sequence of simple reflections such that $s_1 \cdots s_m w = w_0$ and $\ell(s_i \cdots s_m w) < \ell(s_{i-1} \cdots s_m w)$ for each $1 < i < m+1$. That such a sequence exists follows from the uniqueness of w_0 (Prop. 6.1 (i)). Note that Prop. 6.1 (i) also gives that $m = \ell(w_0) - \ell(w) = \ell(ww_0)$. Then

$$\begin{aligned} \mathrm{Ext}^i_{\mathcal{O}_0}(M(w), L(w_0)) &= \mathrm{Ext}^i_{\mathcal{O}_0}(\Theta^!_{s_m} \cdots \Theta^!_{s_1} M(w_0), L(w_0)) \\ &= \mathrm{Ext}^i_{\mathcal{O}_0}(P(w_0), \Theta^*_{s_1} \cdots \Theta^*_{s_m} L(w_0)) \\ &= \mathrm{Ext}^{i-\ell(ww_0)}_{\mathcal{O}_0}(P(w_0), L(w_0)) \\ &= \begin{cases} \mathbb{C} & \text{if } i = \ell(ww_0); \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The first equality is a given by Prop. 7.8 (i), the second equality is by Prop. 6.20 and Prop. 7.4, the third equality is by Prop. 7.8 (iii). $\square$

Remark 7.11. Thm. 7.10 (vi) gives a 'realization' of the Weyl character formula in the Grothendieck group of $\mathcal{O}_0$: $[L(w_0)] = \sum_w (-1)^{\ell(ww_0)}[M(w)]$ (cf. the remarks at the end of [Bott]).

8. Braid relations

For each $w \in W$ fix a reduced word $w = s \cdots t$. Set $\Theta^*_w = \Theta^*_s \cdots \Theta^*_t$ and $\Theta^!_w = \Theta^!_s \cdots \Theta^!_t$. The purpose of this section is to prove that Θ^*_w and $\Theta^!_w$ are independent (up to canonical isomorphism) of the choice of reduced word. More precisely, it will be shown that $\Theta^*_w \Theta^*_{w'} \simeq \Theta^*_{ww'}$ whenever $\ell(ww') = \ell(w) + \ell(w')$ (Prop. 8.6), these are the braid relations. Although this is the most interesting and difficult to obtain property of the functors Θ^*_w, in this document we don't work very hard for it, the result is deduced from the existing literature. I will outline three different methods to obtain Prop. 8.6 using the literature (this is also motivated by a desire to explain the connection of this document with existing work).

In §8.1 I recall some of W. Soergel's results [So90] allowing us to pass from category $\mathcal{O}$ to 'combinatorics'. In this setting, the braid relations are provided by the work of R. Rouquier [Ro, Prop. 3.2]. This section mainly follows [Ro, §4]. I have merely elaborated on some details in [Ro].

The sketch in §8.2 connects the functors $\Theta^!_w$ with R. Irving's shuffling functors [Ir]. This connection seems to have been known by the experts, see Remark 8.8 for the appropriate references. For shuffling functors, the braid relations are known by the work of C. Stroppel and V. Mazorchuk [MazStr07], [MazStr08]. Note that in [MazStr08] and [MazStr08] the braid relations for shuffling functors are proved without ever leaving the context of $\mathfrak{g}$-modules.

The proof sketched in §8.3 uses W. Soergel's functor into combinatorics again. But instead of relying on R. Rouquier's work, I describe W. Soergel's bridge from combinatorics to the geometry of the flag variety [So00]. In this geometric setting the braid relations are essentially a formal consequence of Grothendieck's six functor formalism (actually only three of the functors are ever used in §8.3). The braid relations in this setting seem to have been well known for some time (for instance, see [BBM, §2]). This also explains the connection between this document and [BBM].

8.1. **Proof via Soergel's functor into combinatorics.** Let $S = S(\mathfrak{h})$ denote the algebra of polynomial functons on $\mathfrak{h}^*$. Let $S_+ \subset S$ denote the maximal ideal consisting of functions that vanish at 0. Let $S^W_+ \subset S_+$ be the sub-ideal consisting of W-invariant (regular action) functions in S_+. Let $\mathfrak{z}$ be the center of $U(\mathfrak{g})$ and let $\chi\colon \mathfrak{z} \xrightarrow{\sim} S^{W\cdot}$ be the Harish-Chandra isomorphism (see Thm. 6.4). Note that functions in $S^{W\cdot}$ are W-invariant under the *dot action* while functions in S^W_+ are W-invariant under the *regular action*.

Theorem 8.1 ([So90, Endomorphismensatz 7]). *Let $m\colon \mathfrak{z} \to \mathrm{End}_{\mathfrak{g}}(P(e))$ be the map induced by multiplication. Then*

(i) *m is a surjection $\mathfrak{z} \twoheadrightarrow \mathrm{End}_{\mathfrak{g}}(P(e))$;*

(ii) *the composition* $\mathfrak{z} \xrightarrow{\chi} S^{W\cdot} \hookrightarrow S \twoheadrightarrow S/S_+^W$ *is also a surjection. Here* $S^{W\cdot} \hookrightarrow S$ *is the inclusion map and* $S \twoheadrightarrow S/S_+^W$ *is the projection map;*

(iii) *both* m *and the composite map of (ii) have the same kernel. So we have a unique isomorphism* $\mathrm{End}_{\mathfrak{g}}(P(e)) \xrightarrow{\sim} S/S_+^W$ *that makes the following diagram commute.*

$$\begin{array}{ccccccc}
\mathfrak{z} & & \xrightarrow{\quad m \quad} & & & & \mathrm{End}_{\mathfrak{g}}(P(e)) \\
\Big\| & & & & & & \Big\downarrow \wr \\
\mathfrak{z} & \xrightarrow{\chi} & S^{W\cdot} & \xrightarrow{\text{inclusion}} & S & \xrightarrow{\text{projection}} & S/S_+^W
\end{array}$$

Proof. This was originally proved in [So90]. For an alternate proof see [Ber]. □

The algebra S/S_+^W is the *coinvariant algebra* of W. To ease notation we abbreviate $C = S/S_+^W$. Let $s \in W$ be a simple reflection and let C^s denote the subalgebra of s-invariant elements in C.

Theorem 8.2 ([So90])**.** *Let* $s \in W$ *be a simple reflection. Let* $\nu \in P^+ - \rho$*. Assume that the stabilizer of* ν *under the dot action of* W *is* $\{e, s\}$*. Let* T_ν^0 *be the translation functor* $\mathcal{O}_\nu \to \mathcal{O}_0$ *(see Ch. 7 §7.1). Then* $T_\nu^0 P(w_0 \cdot \nu) \simeq P(e)$*. Furthermore, the induced map* $\mathrm{End}_{\mathfrak{g}}(P(w_0 \cdot \nu)) \to C$ *is an inclusion with image* C^s*.*

Proof. This is [So90, Thm. 8]. Also see [So90, Bemerkung p. 431]. □

Write $C-$fmod (resp. C^s-fmod) for the category of C-modules (resp. C^s-modules). Define

$$\mathbb{V}\colon \mathcal{O}_0 \to C-\mathrm{mod}, \qquad M \mapsto \mathrm{Hom}_{\mathfrak{g}}(P(e), M).$$

Similarly, for $\nu \in P^+ - \rho$ with stabilizer $\{e, s\}$ under the dot action, define

$$\mathbb{V}_\nu\colon \mathcal{O}_\nu \to C^s-\mathrm{mod}, \qquad M \mapsto \mathrm{Hom}_{\mathfrak{g}}(P(w_0 \cdot \nu), M).$$

Proposition 8.3 ([So90])**.** *Let* $\mathcal{O}_0-\mathrm{proj}$ *(resp.* $\mathcal{O}_\nu-\mathrm{proj}$*) be the full subcategory of* $\mathcal{O}_0$ *(resp.* $\mathcal{O}_\nu$*) consisting of projective objects. The functor* $\mathbb{V}\colon \mathcal{O}_0-\mathrm{proj} \to C-\mathrm{mod}$ *(resp.* $\mathbb{V}_\nu\colon \mathcal{O}_\nu-\mathrm{proj} \to C^s-\mathrm{mod}$*) is full and faithful.*

Proof. This is [So90, Struktursatz 9]. □

Let $s \in W$ be a simple reflection. Let $\mathrm{Ind}_s\colon C^s-\mathrm{mod} \to C-\mathrm{mod}$ be the induction functor, i.e., $\mathrm{Ind}_s = C \otimes_{C^s} -$. Let $\mathrm{Res}_s\colon C-\mathrm{mod} \to C^s-\mathrm{mod}$ be the restriction functor, i.e., $\mathrm{Res}_s(M)$ is just M viewed as a C^s-module. Define $\tilde{\varepsilon}\colon \mathrm{Ind}_s\mathrm{Res}_s \to \mathrm{id}$ by $\tilde{\varepsilon}_M\colon C \otimes_{C^s} M \to M$, $\tilde{\varepsilon}_M\colon c \otimes m \mapsto cm$. Define $\tilde{\eta}\colon \mathrm{id} \to \mathrm{Res}_s\mathrm{Ind}_s$ by $\tilde{\eta}_M\colon M \to C \otimes_{C^s} M$, $\tilde{\eta}_M\colon m \mapsto 1 \otimes m$. This defines an adjunction $(\mathrm{Ind}_s, \mathrm{Res}_s)$.

Let F_s be the complex of functors

$$F_s = 0 \to \mathrm{Ind}_s\mathrm{Res}_s \xrightarrow{\tilde{\varepsilon}} \mathrm{id} \to 0$$

with $\mathrm{Ind}_s\mathrm{Res}_s$ in degree 0. For each $w \in W$ fix a reduced word $w = r \cdots t$. Set $F_w = F_r \cdots F_t$.

Proposition 8.4 ([Ro])**.** *Let* $w, w' \in W$*. If* $\ell(ww') = \ell(w) + \ell(w')$*, then* $F_w F_{w'} \simeq F_{ww'}$ *as functors on* $\mathrm{Ho}^{\mathrm{b}}(C-\mathrm{mod})$*.*

Proof. See [Ro, Prop. 3.2] and [Ro, §3.3.3]. □

Proposition 8.5 ([So90]). *Let $s \in W$ be a simple reflection. Let $\nu \in P^+ - \rho$. Assume that the stabilizer of ν under the dot action of W is $\{e, s\}$. Let $\pi_{s*} = T_0^\nu$ and $\pi_s^* = T_\nu^0$ be the functors of translation on the s-wall and off the s-wall, respectively (see Ch. 7 §7.2). Then we have canonical isomorphisms*

$$\mathbb{V}_\nu \circ \pi_{s*} \simeq \mathrm{Res}_s \circ \mathbb{V} \qquad \textit{and} \qquad \mathbb{V} \circ \pi_s^* \simeq \mathrm{Ind}_s \circ \mathbb{V}_\nu.$$

Proof. This is deduced by combining Thm. 8.2 with the fact that π_{s*} and π_s^* are left and right adjoint to each other. See [So90, Thm. 10] for details. □

Except for some technical bookkeeping, it is now intuitively clear that

Proposition 8.6. *Let $w, w' \in W$. If $\ell(ww') = \ell(w) + \ell(w')$, then $\Theta_w^* \Theta_{w'}^* \simeq \Theta_{ww'}^*$.*

Proof. Let $s \in W$ be a simple reflection. Let $\nu \in P^+ - \rho$ be such that the stabilizer of ν under the dot action is $\{e, s\}$. Let $\mathcal{C}$ (resp. $\mathcal{C}^s$) be the full subcategory of $C-$mod (resp. C^s-mod) consisting of objects that are isomorphic to $\mathbb{V}P$ (resp. $\mathbb{V}_\nu P$) for some $P \in \mathcal{O}_0-$proj (resp. $P \in \mathcal{O}_\nu-$proj). By Prop. 1.1, $\mathbb{V}\colon \mathcal{O}_0-\mathrm{proj} \to \mathcal{C}$ and $\mathbb{V}_\nu\colon \mathcal{O}_\nu-\mathrm{proj} \to \mathcal{C}^s$ are equivalences. Further, π_{s*} and π_s^* restrict to functors $\mathcal{O}_0-\mathrm{proj} \to \mathcal{O}_\nu-\mathrm{proj}$ and $\mathcal{O}_\nu-\mathrm{proj} \to \mathcal{O}_0-\mathrm{proj}$, since both π_{s*} and π_s^* are left adjoint to exact functors (see Remark 3.15). So, it follows from Prop. 8.5 that Res_s and Ind_s restrict to functors $\mathcal{C} \to \mathcal{C}^s$ and $\mathcal{C}^s \to \mathcal{C}$, respectively.

By Prop. 8.5 there is an isomorphism $\phi\colon \pi_{s*} \xrightarrow{\sim} \mathbb{V}_\nu^{-1} \circ \mathrm{Res}_s \circ \mathbb{V}$ of functors $\mathcal{O}_0-\mathrm{proj} \to \mathcal{O}_\nu-\mathrm{proj}$. Fix an adjunction (π_s^*, π_{s*}) with counit ε and an adjunction $(\mathrm{Ind}_s, \mathrm{Res}_s\mathbb{V})$ with unit $\tilde{\eta}$ and counit $\tilde{\varepsilon}$. Let $a\colon \mathbb{V}^{-1}\mathbb{V} \xrightarrow{\sim} \mathrm{id}$ and $b\colon \mathbb{V}_\nu \mathbb{V}_\nu^{-1} \xrightarrow{\sim} \mathrm{id}$ be fixed isomorphisms. Set $\tilde{\eta}' = \mathbb{1}_{\mathbb{V}^{-1}\mathrm{Ind}_s} b^{-1} \mathbb{1}_{\mathrm{Res}_s\mathbb{V}} \circ \mathbb{1}_{\mathbb{V}^{-1}} \tilde{\eta} \mathbb{1}_{\mathbb{V}} \circ a^{-1}$ and $\tilde{\varepsilon}' = a \circ \mathbb{1}_{\mathbb{V}^{-1}} \tilde{\varepsilon} \mathbb{1}_{\mathbb{V}} \circ \mathbb{1}_{\mathbb{V}^{-1}\mathrm{Ind}_s} b \mathbb{1}_{\mathrm{Res}_s\mathbb{V}}$. Then $\tilde{\eta}'$ and $\tilde{\varepsilon}'$ define an adjunction $(\mathbb{V}^{-1}\mathrm{Ind}_s\mathbb{V}_\nu, \mathbb{V}_\nu^{-1}\mathrm{Res}_s\mathbb{V})$. Let $\psi\colon \pi_s^* \xrightarrow{\sim} \mathbb{V}^{-1} \circ \mathrm{Ind}_s \circ \mathbb{V}_\nu$ be the transpose of ϕ^{-1} (see Ch. 2 §2.2). Let $\gamma = \mathbb{1}_{\mathbb{V}^{-1}\mathrm{Ind}_s} b \mathbb{1}_{\mathrm{Res}_s\mathbb{V}} \circ \psi \mathbb{1}_{\mathbb{V}_\nu^{-1}\mathrm{Res}_s\mathbb{V}} \circ \mathbb{1}_{\pi_s^*}\phi$. Then by Prop. 2.3 (i), $a^{-1} \circ \varepsilon = \mathbb{1}_{\mathbb{V}^{-1}} \tilde{\varepsilon} \mathbb{1}_{\mathbb{V}} \circ \gamma$. Further, Prop. 2.3 (iii) and Prop. 2.4 (i) imply that γ is an isomorphism. Now work in the 2-category $\mathfrak{K}$ of Ch. 4 §4.4. In particular, work in the triangulated category $\mathscr{H}om_{\mathfrak{K}}(\mathrm{Ho}^{\mathrm{b}}(\mathcal{O}_0-\mathrm{proj}), \mathrm{Ho}^{\mathrm{b}}(\mathcal{O}_0-\mathrm{proj}))$. By Thm. 4.5, the cocone of $\mathbb{1}_{\mathbb{V}^{-1}} \tilde{\varepsilon} \mathbb{1}_{\mathbb{V}}$ is isomorphic to $\mathbb{V}^{-1}F_s\mathbb{V}$. So we have a commutative diagram

$$\begin{array}{ccccccc}
\Theta_s^* & \longrightarrow & \pi_s^*\pi_{s*} & \xrightarrow{\varepsilon} & \mathrm{id} & \rightsquigarrow & \\
 & & \gamma \downarrow \wr & & a^{-1} \downarrow \wr & & \\
\mathbb{V}^{-1}F_s\mathbb{V} & \longrightarrow & \mathbb{V}^{-1}\mathrm{Ind}_s\mathrm{Res}_s\mathbb{V} & \xrightarrow{\mathbb{1}_{\mathbb{V}^{-1}}\tilde{\varepsilon}\mathbb{1}_{\mathbb{V}}} & \mathbb{V}^{-1}\mathbb{V} & \rightsquigarrow &
\end{array}$$

in which the rows are distinguished triangles. This diagram completes to an isomorphism $\Theta_s^* \simeq F_s'$ (see [KaSc06, Prop. 10.1.15]). Thus, Prop. 8.4 implies that if $\ell(ww') = \ell(w) + \ell(w')$, then $\Theta_w^* \Theta_{w'}^* \simeq \Theta_{ww'}^*$ in $\mathscr{H}om_{\mathfrak{K}}(\mathrm{Ho}^{\mathrm{b}}(\mathcal{O}_0-\mathrm{proj}), \mathrm{Ho}^{\mathrm{b}}(\mathcal{O}_0-\mathrm{proj}))$. From Prop. 6.22, we deduce that every object in $\mathrm{D}^{\mathrm{b}}(\mathcal{O}_0)$ is quasisomorphic to a bounded complex of projectives. This gives the result. □

Remark 8.7. For our purposes, i.e., the applications in Ch. 9, all that is needed is that if $X \in \mathrm{D}^{\mathrm{b}}(\mathcal{O}_0)$, then $\Theta_w^* \Theta_{w'}^* X \simeq \Theta_{ww'}^* X$ whenever $\ell(ww') = \ell(w) + \ell(w')$. This can be shown by an argument similar to the proof of Prop. 8.6 without requiring the technicalities of the 2-category $\mathfrak{K}$. However, this would make the arguments and language in the sequel cumbersome.

8.2. **Sketch of a proof via shuffling functors.** Let $X \in \mathcal{O}_0$. Set $C_s X = \operatorname{coker}(\eta' \colon X \to \pi_s^* \pi_{s*} X)$. As cokernels in abelian categories are unique up to *canonical* isomorphism (see [KaSc06, §8.3]), this gives a well defined functor $\mathcal{O}_0 \to \mathcal{O}_0$ along with a canonical morphism $c \colon \pi_s^* \pi_{s*} \to C_s$ such that $\pi_s^* \pi_{s*} X \xrightarrow{\eta'_X} X \to C_s X \to 0$ is exact for each $X \in \mathcal{O}_0$. The functor C_s is right exact (cf. [KaSc06, Exer. 13.23]). These are in fact very general statements that hold for any adjoint pair of exact functors between abelian categories. In the context of category $\mathcal{O}_0$ the functor C_s was first considered in [Ir] and is referred to as the *shuffling* functor.

Define $f \colon \Theta_s^! \to C_s$ by

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_s^* \pi_{s*} & \xrightarrow{\eta'} & \mathrm{id} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle c} & & \downarrow & & \\ 0 & \longrightarrow & C_s & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

(cf. [KaSc06, Cor. 12.3.5]). Let $P \in \mathrm{Comp}^{\mathrm{b}}(\mathcal{O}_0\mathrm{-proj})$. Combining Prop. 7.6 (i) with Prop. 6.21 we deduce that $0 \to P \xrightarrow{\eta'_P} \pi_s^* \pi_{s*} P \xrightarrow{c_P} C_s P \to 0$ is an exact sequence in $\mathrm{Comp}^{\mathrm{b}}(\mathcal{O}_0)$. Let $\mathrm{quot}\mathrm{Ho}^{\mathrm{b}}(\mathcal{O}_0) \to \mathrm{D}^{\mathrm{b}}(\mathcal{O}_0)$ be the localization functor (see Ch. 3 §3.4). Then it follows that f is an isomorphism, of functors $\mathrm{Ho}^{\mathrm{b}}(\mathcal{O}_0\mathrm{-proj}) \to \mathrm{D}^{\mathrm{b}}(\mathcal{O}_0)$, of $\Theta_s^!$ with the composition $\mathrm{Ho}^{\mathrm{b}}(\mathcal{O}_0\mathrm{-proj}) \xrightarrow{C_s} \mathrm{Ho}^{\mathrm{b}}(\mathcal{O}_0) \xrightarrow{\mathrm{quot}} \mathrm{D}^{\mathrm{b}}(\mathcal{O}_0)$ (cf. [KaSc06, Prop. 13.1.13]). Let $\mathbf{L}C_s$ denote the left derived functor of C_s. This is the composition $\mathrm{D}^{\mathrm{b}}(\mathcal{O}_0) \xrightarrow{\mathbf{p}} \mathrm{Ho}^{\mathrm{b}}(\mathcal{O}_0\mathrm{-proj}) \xrightarrow{C_s} \mathrm{Ho}^{\mathrm{b}}(\mathcal{O}_0) \xrightarrow{\mathrm{quot}} \mathrm{D}^{\mathrm{b}}(\mathcal{O}_0)$, where, for $X \in \mathrm{D}^{\mathrm{b}}(\mathcal{O}_0)$, $\mathbf{p}X$ is a projective resolution of X (see (3.1) and Prop. 6.21). Thus, $\Theta_s^! \simeq \mathbf{L}C_s$.

Remark 8.8. That $\Theta_s^! \simeq \mathbf{L}C_s$ seems to have been known to the experts for a while (see [AL, Remark 1.2], [MazStr08, Remark 5.8] and [Ro, Remark 4.6]). It is also (essentially) Exer. 13.19 and Exer. 13.23 in [KaSc06].

For each $w \in W$ fix a reduced word $w = s \cdots t$, set $C_w = C_s \cdots C_t$.

Proposition 8.9. *Let $w, w' \in W$. If $\ell(ww') = \ell(w) + \ell(w')$, then $C_w C_{w'} \simeq C_{ww'}$.*

Proof. This is [MazStr08, Lemma 5.10]. □

Proposition 8.10. *Let $w \in W$. If $w = s \cdots t$ is a reduced word, then $\mathbf{L}C_s \cdots \mathbf{L}C_t \simeq \mathbf{L}(C_s \cdots C_t)$.*

Proof. This is [MazStr07, Prop. 10.1]. □

It now follows that if $w, w' \in W$ is such that $\ell(ww') = \ell(w) + \ell(w')$, then

$$\Theta_w^! \Theta_{w'}^! \simeq \mathbf{L}C_w \mathbf{L}C_{w'} \simeq \mathbf{L}(C_w C'_w) \simeq \mathbf{L}C_{ww'} \simeq \Theta_{ww'}^!.$$

8.3. **Sketch of a proof via correspondences on the flag variety.** Let X be a complex algebraic variety. Let $\mathrm{Sh}(X)$ denote the category of sheaves of $\mathbb{C}$-vector spaces on X (for generalities on sheaves see [KaSc90, Ch. II, III]). Let $f \colon X \to Y$ be a morphism of varieties. Let $f^* \colon \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$, $f_* \colon \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ and $f_! \colon \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ denote the functors of pullback, pushforward and pushforward with proper supports, respectively. Then f^* is exact and $f_*, f_!$ are left exact. Further, f^* is left adjoint to f_*. There is always a canonical morphism $f_! \to f_*$

which is an isomorphism whenever f is proper. Let pt be the single point space. Let $\underline{\mathbb{C}}_{\mathrm{pt}}$ be the sheaf $\mathbb{C}$ on pt. Let $a\colon X \to \mathrm{pt}$ be the constant map. Then the constant sheaf on X is $\underline{\mathbb{C}}_X = a^*\underline{\mathbb{C}}_{\mathrm{pt}}$.

Let $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$ denote the bounded derived category of cohomologically constructible sheaves of $\mathbb{C}$-vector spaces on X. Let $\mathbf{R}f_*\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(Y)$ and $\mathbf{R}f_!\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(Y)$ denote the right derived functors of f_* and $f_!$, respectively.

These functors enjoy a number of useful properties, we list two of them. Much of what is to follow is just formal manipulations using these properties.

- [KaSc90, (2.3.8), (2.3.9), (2.5.3)] Given a sequence of maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, there are canonical isomorphisms
$$(gf)^* \simeq f^*g^*, \qquad (gf)_* \simeq g_*f_* \qquad \text{and} \qquad (gf)_! \simeq g_!f_!.$$
- *Proper base change* [KaSc90, Prop. 2.5.11]: Given a cartesian diagram
$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{\tilde{f}} & Y \\ {\scriptstyle \tilde{g}}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z \end{array}$$
there is a canonical isomorphism of functors
$$g^*f_! \simeq \tilde{f}_!\tilde{g}^*.$$

These properties also transfer to the derived versions of these functors (see [KaSc90, (2.6.5), (2.6.6), Prop. 2.6.7])

8.4. **The flag variety.** Let G be a complex, simple, algebraic group with Lie algebra $\mathfrak{g}$. For generalities on algebraic groups see [Bo91, Ch. 1]. Let $B \subset G$ be a Borel subgroup with Lie algebra $\mathfrak{b}$. Let $T \subset B$ be a maximal torus. Write $N_G(T)$ for the normalizer of T in G. Then we have an isomorphism $W \simeq N_G(T)/T$ (see [Bo91, Ch. IV §11]). The Bruhat decoomposition states that $G = \bigsqcup_{w \in W} B\dot{w}B$, where $\dot{w}$ is a representative of $w \in N_G(T)$ [Bo91, Ch. IV, Thm. 14.12 (a)]. Let G/B be the flag variety (see [Bo91, Ch. IV §11.18]). This is a smooth projective variety [Bo91, Ch. IV, Thm. 11.1]. The Bruhat decomposition gives a decomposition $G/B = \bigsqcup w \in WB\dot{w}B/B$ [Bo91, Ch. IV, Thm. 14.12(b)].

For $w \in W$, let
$$Z_w = G\text{-diagonal orbit of } (B/B, \dot{w}B/B) \text{ in } G/B \times G/B.$$
Then $G/B \times G/B = \bigsqcup_{w \in W} Z_w$ [CG, Prop. 3.1.29]. Let $p_w\colon Z_w \to G/B$ and $q_w\colon Z_w \to G/B$ be the projection on the first and second factor. This gives a functor
$$R^*_w = \mathbf{R}q_{w!}p^*_w[-\ell(w)]\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(G/B) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(G/B).$$
The functors R^*_w go by several names in the literature: Radon transforms [BBM], intertwining functors [ABG], Hecke correspondences [Bei], etc. We will refer to them as correspondences on G/B.

Let $w, w' \in W$ and let $Z_w \times_{G/B} Z_{w'}$ be the fiber product of Z_w and Z'_w with respect to the morphisms p_w and $q_{w'}$. Let $\alpha_{w'}\colon Z_w \times_{G/B} Z_{w'} \to Z_{w'}$ and $\beta_w\colon Z_w \times_{\mathcal{B}}$

$Z_{w'} \to Z_w$ be the corresponding projection maps. Then we obtain a commutative diagram

(8.1)
$$\begin{array}{ccccc} Z_w \times_{G/B} Z_{w'} & \xrightarrow{\beta_w} & Z_w & \xrightarrow{q_w} & G/B \\ {\scriptstyle \alpha_{w'}}\downarrow & & \downarrow{\scriptstyle p_w} & & \\ Z_{w'} & \xrightarrow{q_{w'}} & G/B & & \\ {\scriptstyle p_{w'}}\downarrow & & & & \\ G/B & & & & \end{array}$$

The compositions $p_{w'} \circ \alpha_{w'} \colon Z_w \times_{G/B} Z_{w'} \to G/B$ and $q_w \circ \beta_w \colon Z_w \times_{G/B} Z_{w'} \to G/B$ determine a unique morphism $r_{w,w'} \colon Z_w \times_{G/B} Z_{w'} \to G/B \times G/B$ such that the following diagram commutes:

(8.2)
$$\begin{array}{ccccc} & & Z_w \times_{G/B} Z_{w'} & & \\ & \swarrow^{p_{w'}\alpha_{w'}} & \downarrow{\scriptstyle r} & {}^{q_w\beta_w}\searrow & \\ & & G/B \times G/B & & \\ & \swarrow^{p} & & {}^{q}\searrow & \\ G/B & & & & G/B \end{array}$$

where p and q are projection on the first and second factor, respectively.

The following is well known.

Proposition 8.11. *The image of $r_{w,w'}$ is contained in $Z_{ww'}$. If $\ell(ww') = \ell(w) + \ell(w')$, then $r_{w,w'} \colon Z_w \times_{G/B} Z_{w'} \to Z_{ww'}$ is an isomorphism.*

Proof. See [Mil, §L.3, Lemma 6]. □

Corollary 8.12. *Let $w, w' \in W$. If $\ell(ww') = \ell(w) + \ell(w')$, then $R_w^* R_{w'}^* \simeq R_{ww'}^*$.*

Proof. We have

$$\begin{aligned} R_w^* R_{w'}^* &= \mathbf{R}q_{w!}p_w^* \mathbf{R}q_{w'!}p_{w'}^*[-\ell(ww')] \\ &\simeq \mathbf{R}q_{w!}\mathbf{R}\beta_{w!}\alpha_{w'}^* p_{w'}^*[-\ell(ww')] \qquad \text{(by (8.1) and proper base change)} \\ &\simeq \mathbf{R}(q_w\beta_w)_!(p_{w'}\alpha_{w'})^*[-\ell(ww')] \\ &= \mathbf{R}(qr)_!(pr)^*[-\ell(ww')] \qquad \text{(by (8.2))} \\ &= \mathbf{R}p_{ww'!}q_{ww'}^*[-\ell(ww')] \qquad \text{(by Prop. 8.11)} \\ &= R_{ww'}^*. \end{aligned}$$

□

Let $s \in W$ be a simple reflection. Let $P_s \supset B$ be the minimal parabolic subgroup corresponding to s. That is $P_s = B \sqcup B\dot{s}B$. For details see [Bo91, Ch. IV §11.2].

Proposition 8.13. *The projection $\tilde{\pi}_s \colon G/B \to G/P_s$ is locally trivial. Its fibers are isomorphic to the projective line $\mathbb{P}^1$. In particular, $\tilde{\pi}_s$ is smooth and proper.*

Proof. See [Mil, §L.3, Lemma 10]. □

Proposition 8.14. *Let $s \in W$ be a simple reflecton. Then the following diagram is cartesian.*

$$\begin{array}{ccc} \overline{Z}_s & \xrightarrow{\overline{q}_s} & G/B \\ {\scriptstyle \overline{p}_s}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}_s} \\ G/B & \xrightarrow{\tilde{\pi}_s} & G/P_s \end{array}$$

where $\overline{Z}_s$ is the closure of Z_s in $G/B \times G/B$ and $\overline{p}_s, \overline{q}_s \colon \overline{Z}_s \to G/B$ are projection on the first and second factor, respectively. Furthermore, $\overline{Z}_s = Z_e \sqcup Z_s$.

Proof. That $\overline{Z}_s = Z_e \sqcup Z_s$ can be found in [Sp97, §1.5]. With this in hand, the fact that $\overline{Z} = G/B \times_{G/P} G/B$ follows from the Bruhat decomposition and the fact that $P = B \sqcup B\dot{s}B$. For details, see [Mil, §L.7]. □

Fix an adjunction $(\tilde{\pi}_s^*, \tilde{\pi}_{s*})$, this gives an adjunction $(\tilde{\pi}_s^*, \mathbf{R}\tilde{\pi}_{s*})$. Let ε be the counit of this adjunction. The following result should be read in the context of the 2-category $\mathfrak{K}$ of Ch. 4.

Proposition 8.15. *Let $s \in W$ be a simple reflection. Then R_s^* is isomorphic to the cocone of $\varepsilon \colon \tilde{\pi}_s^* \mathbf{R}\tilde{\pi}_{s*} \to \mathrm{id}$.*

Sketch of proof. This follows from Prop. 8.14, Prop. 8.13, proper base change and the adjunction triangle (see [Dim, §2.4]). For further details see [Ro, §2.2.1]. Also see the comments after Lemma 7 in [Mil]. □

Proposition 8.16. *The cohomology ring $H^\bullet(G/B)$ is isomorphic to the coinvariant algebra C (see §8.1). Further, the projection $\tilde{\pi}_s \colon G/B \to G/P$ induces an injection $H^\bullet(G/P_s) \hookrightarrow H^\bullet(G/B)$. The image of this injection is C^s.*

Proof. See [Bo53, §27] and [BGG73, §1.3]. Also cf. [Dem]. □

The cohomology ring may also be interpreted as the endomorphism algebra of the constant sheaf in the derived category (see [CG, §8.3]). That is $H^\bullet(G/B) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(G/B)}(\underline{\mathbb{C}}_{G/B}, \underline{\mathbb{C}}_{G/B}[n])$. Define

$$\mathbb{H} \colon \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(G/B) \to C\text{-mod}, \qquad \mathcal{F} \mapsto \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(G/B)}(\underline{\mathbb{C}}_{G/B}, \mathcal{F}[n]).$$

Similarly, define

$$\mathbb{H}_s \colon \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(G/P) \to C^s\text{-mod}, \qquad \mathcal{F} \mapsto \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(G/P)}(\underline{\mathbb{C}}_{G/P}, \mathcal{F}[n])$$

Let $\mathcal{K}$ be the smallest full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(G/B)$ satisfying the following:

- if $\mathcal{F} \in \mathcal{K}$ then all objects isomorphic to $\mathcal{F}$ are also in $\mathcal{K}$;
- $i_* \underline{\mathbb{C}}_{B/B} \in \mathcal{K}$, where $i \colon B/B \hookrightarrow G/B$ is the inclusion of the point B/B;
- if $\mathcal{F} \in \mathcal{K}$, then $\mathcal{F}[n] \in \mathcal{K}$;
- if $\mathcal{F} \in \mathcal{K}$, then $\tilde{\pi}_s^* \tilde{\pi}_{s*} \mathcal{F} \in \mathcal{K}$ for each simple reflection $s \in W$;
- if $\mathcal{F}, \mathcal{E} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(G/B)$, then $\mathcal{F}, \mathcal{E} \in \mathcal{K}$ if and only if $\mathcal{F} \oplus \mathcal{E} \in \mathcal{K}$.

Let $s \in W$ be a simple reflection. Let $\mathcal{K}^s$ be the smallest full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(G/P)$ satisfying the same properties as above (with the obvious modifications). Let $\nu \in P^+ - \rho$ be such that the stabilizer of ν under the dot action of W is $\{e, s\}$. Let $\mathcal{C}$ (resp. $\mathcal{C}^s$) be the full subcategory of C-mod (resp. C^s-mod) consisting of objects that are isomorphic to $\mathbb{V}P$ (resp. $\mathbb{V}_\nu P$) for some $P \in \mathcal{O}_0-$proj (resp. $\mathcal{O}_\nu-$proj).

Theorem 8.17 ([So00]). *The functors $\mathbb{H}\colon \mathcal{K} \to \mathcal{C}$ and $\mathbb{H}_s\colon \mathcal{K}^s \to \mathcal{C}^s$ are equivalences of categories. Furthermore, there are canonical isomorphisms*

$$\mathbb{H}_s \circ \mathbf{R}\tilde{\pi}_{s*} \simeq \mathrm{Res}_s \circ \mathbb{H} \qquad \text{and} \qquad \mathbb{H} \circ \tilde{\pi}_s^*. \simeq \mathrm{Ind}_s \circ \mathbb{H}_s,$$

where $\mathrm{Ind}_s\colon \mathcal{C}^s \to \mathcal{C}$ and $\mathrm{Res}_s\colon \mathcal{C} \to \mathcal{C}^s$ are the induction and restriction functors, respectively (see §8.1).

Proof. The first statement is [So00, Thm. 4.2.1] (cf. [BGS, Thm. 3.7.1], [So90, Erweiterungssatz 17]). The second statement is [So00, Prop. 4.1.1] (also see Prop. 8.16). □

Now combining Thm. 8.17 with Cor. 8.12 and Prop. 8.15 gives Prop. 8.4.

9. Ringel self-duality of $\mathcal{O}_0$ and Soergel's character formula

The goal of this section is to prove Thm. 9.6. Our approach mimics that of [BBM, Prop. 2.3]. For the connection between this work and [BBM] see Ch. 8 §8.3. Similar results have been obtained in the setting of D-modules on the flag variety [BeGi, Thm. 6.10]. The ideas in our approach are also implicit in [Maz03], [MazStr05, §5.4] and [MazStr08, Prop. 4.1] (cf. Ch. 8 §8.2).

Proposition 9.1. *Let $w \in W$.*
- (i) $\Theta^*_w M(e) \simeq M(w)$.
- (ii) $\Theta^!_w M(e) \simeq M(w)^\vee$.

Proof. Let $w = s \cdots t$ be a reduced word. Then $\Theta^*_w \simeq \Theta^*_s \cdots \Theta^*_t$ by Prop. 8.6. Hence, by Prop. 7.8 (i),

$$\Theta^*_w M(e) \simeq \Theta^*_s \cdots \Theta^*_t M(e) \simeq M(s \cdots t) = M(w).$$

This proves (i). The proof of (ii) is analogous (note that $M(e) = M(e)^\vee$). □

Lemma 9.2. *Let $w \in W$ and let w_0 be the longest element in W.*
- (i) $\Theta^*_{w_0} M(w)^\vee \simeq M(w_0 w)$.
- (ii) $\Theta^!_{w_0} M(w) \simeq M(w_0 w)^\vee$.

Proof. We have

$$\Theta^*_{w_0} M(w)^\vee \simeq \Theta^*_{w_0}\Theta^!_w M(e) \simeq \Theta^*_{w_0 w}\Theta^*_{w^{-1}}\Theta^!_w M(e) \simeq \Theta^*_{w_0 w} M(e) \simeq M(w_0 w).$$

The first isomorphism is Prop. 9.1 (ii), the second isomorphism follows from Prop. 8.6, the third isomorphism follows from Prop. 7.4 and the last isomorphism is Prop. 9.1 (i). This proves (i). For (ii), note that the uniqueness of w_0 (Prop. 6.1 (i)) implies that $w_0^{-1} = w_0$. Combining this with Prop. 7.4 we deduce that $(\Theta^*_{w_0})^{-1} = \Theta^!_{w_0}$. Thus, (ii) follows from (i). □

We ask the reader to recall the notion of filtrations in triangulated categories and some of the notation in Ch. 3 §3.1. In particular, if $\mathcal{A}$ is a subcategory of $\mathrm{D^b}(\mathcal{O})$, then

$$\mathcal{A} * \mathcal{A} = \{Y \in \mathrm{D^b}(\mathcal{O}) \mid \text{there exists } X \to Y \to Z \rightsquigarrow, \text{ with } [X], [Z] \in \mathcal{A}\},$$

where $[X]$ denotes the isomorphism class of X. Further, recall that $\mathcal{A}^{*i} = \mathcal{A} * \mathcal{A}^{*i-1}$, and that $\mathcal{A}^{*\infty} = \bigcup_{i \in \mathbb{Z}_{\geq 0}} \mathcal{A}^{*i}$.

Proposition 9.3. *Let $X \in \mathcal{O}_0$, then, as an object of $\mathrm{D^b}(\mathcal{O}_0)$, X is filtered by objects of the form $M(w)[i]$, $i \geq 0$, $w \in W$.*

Proof. Let $\mathcal{O}_\Delta$ be the subcategory of $\mathrm{D^b}(\mathcal{O}_0)$ consisting of objects $M(w)[i]$, $i \in \mathbb{Z}_{\geq 0}$, $w \in W$. Note that if $M \in \mathcal{O}_\Delta^{*\infty}$, then $M[i] \in \mathcal{O}_\Delta^{*\infty}$ for all $i \in \mathbb{Z}_{\geq 0}$. By Lemma 3.3, it suffices to show that $\mathcal{O}_0 \subset \mathcal{O}_\Delta^{*\infty}$. Since every object in $\mathcal{O}_0$ has finite length (Prop. 6.10), this reduces to showing that each $L(w)$, $w \in W$, is in $\mathcal{O}_\Delta^{*\infty}$. Proceed by induction on the length of w. If $\ell(w) = 0$, then $w = e$ and $L(w) = L(e) = M(e)$ which is clearly in $\mathcal{O}_\Delta^{*\infty}$. Now let $w \in W$ and assume that if $\ell(w') < \ell(w)$, then $L(w') \in \mathcal{O}_\Delta^{*\infty}$. Let $N(w)$ be the kernel of the map $M(w) \twoheadrightarrow L(w)$. Then the exact sequence $0 \to N(w) \to M(w) \to L(w) \to 0$ gives a distinguished triangle $M(w) \to L(w) \to N(w)[1] \rightsquigarrow$ in $\mathrm{D^b}(\mathcal{O}_0)$ (see Prop. 3.8). By the induction hypothesis $N(w)[1] \in \mathcal{O}_\Delta^{*\infty}$. Consequently, $L(w) \in \mathcal{O}_\Delta^{*\infty}$. □

The following result is the category $\mathcal{O}$ analogue of [BeGi, Thm. 6.10] (D-modules) and [BBM, §2.3] (perverse sheaves). In fact the proof presented here mimics the proof of [BBM, Prop. 2.3] (see §8.3 for a more precise connection).

Theorem 9.4. *Let $w \in W$ and let w_0 be the longest element in W. Then*
- (i) $\Theta^*_{w_0} T(w) \simeq P(w_0 w)$;
- (ii) $\Theta^*_{w_0} I(w) \simeq T(w_0 w)$.

Proof. We will only prove (i), the proof of (ii) is similar. Since $T(w)$ has a dual Verma filtration, Prop. 7.7 (ii) implies that $\Theta^*_{w_0} T(w)$ lies in $\mathcal{O}_0$. Let $w' \in W$ and let $i > 0$, then

$$\mathrm{Ext}^i_{\mathcal{O}_0}(\Theta^*_{w_0} T(w), M(w')) = \mathrm{Ext}^i_{\mathcal{O}_0}(T(w), \Theta^!_{w_0} M(w')) = \mathrm{Ext}^i_{\mathcal{O}_0}(T(w), M(w_0 w')^\vee) = 0.$$

The first equality is given by Prop. 7.4, Prop. 8.6 and the observation that the uniqueness of w_0 (Prop. 6.1 (i)) implies $w_0^{-1} = w_0$. The second equality is Lemma 9.2 (ii) and the last equality is given by Prop. 6.23. Combining this with Lemma 9.3 we deduce that if $i > 0$, then $\mathrm{Ext}^i_{\mathcal{O}_0}(\Theta^*_{w_0} T(w), X) = 0$ for all $X \in \mathcal{O}_0$ (see Remark 3.4). Thus $\Theta^*_{w_0} T(w)$ is projective. Since $T(w)$ is indecomposable and $\Theta^*_{w_0}$ is an equivalence (Prop. 7.4), we deduce that $\Theta^*_{w_0} T(w)$ is also indecomposable. It remains to show that $\Theta^*_{w_0} T(w)$ surjects onto $L(w_0 w)$. Lemma 9.2 (i) implies that $\Theta^*_{w_0} T(w)$ has a Verma filtration $0 \subset V_1 \subset \cdots \subset V_{m+1} = \Theta_{w_0} T(w)$, with $V_{m+1}/V_m \simeq M(w_0 w)$. Thus, $\Theta_{w_0} T(w)$ surjects onto $L(w_0 w)$. □

Corollary 9.5 ([So98, Thm. 6.7])**.** *Let $w, w' \in W$ and let w_0 be the longest element in W. Then, at the level of the Grothendieck group $K_0(\mathcal{O}_0)$:*

$$[T(w) : M(w')] = [P(w_0 w) : M(w_0 w')] = [M(w_0 w') : L(w_0 w)].$$

Proof. Working in $K_0(\mathcal{O}_0)$, we have

$$[T(w) : M(w')] = [\Theta^*_{w_0} T(w) : \Theta^*_{w_0} M(w')] = [P(w_0 w) : M(w_0 w')^\vee] = [M(w_0 w') : L(w_0 w)].$$

The first equality is a consequence of Prop. 7.4. The second equality is obtained from Thm. 9.6 (i) and by combining Lemma 9.2 (ii) with the fact that at the level of of $K_0(\mathcal{O}_0)$, $[\Theta^*_s X] = [\Theta^!_s X]$ for all $X \in \mathcal{O}_0$ and each simple reflection $s \in W$ (see Remark 7.5). The last equality is BGG reciprocity (Prop. 6.25). □

Corollary 9.6 ([So98])**.** $\bigoplus_{w \in W} \mathrm{End}(P(w)) \simeq \bigoplus_{w \in W} \mathrm{End}(T(w))$.

Proof. Let w_0 be the longest element in W. Then by the uniqueness of w_0 (Prop. 6.1 (i)) we have $w_0^{-1} = w_0$. Thus, Prop. 7.4 gives that $(\Theta^*_{w_0})^{-1} = \Theta^!_{w_0}$. So, by Thm. 9.6 (i), we have

$$\bigoplus_{w\in W} \operatorname{End}(P(w)) \simeq \bigoplus_{w\in W} \operatorname{End}(\Theta^!_{w_0} P(w)) \simeq \bigoplus_{w\in W} \operatorname{End}(T(w)) \qquad \square$$

References

[And] H. H. Andersen, *Filtrations and tilting modules*, Ann. Sci. École Norm. Sup. (4) **30** (1997), no. 3, 353-366.

[AL] H. H. Andersen, N. Lauritzen, *Twisted Verma modules*, Studies in memory of Issai Schur, 1-26, Progr. Math. **210**, Birkhäuser Boston, Boston, MA (2003).

[Ark] S. Arkhipov, *Semi-infinite cohomology of associative algebras and bar duality*, Internat. Math. Res. Notices (1997), no. 17, 833-863.

[ABG] S. Arkhipov, R. Bezrukavnikov, V. Ginzburg, *Quantum groups, the loop Grassmannian, and the Springer resolution*, J. Amer. Math. Soc. **17**, no. 3 (2004), 595-678.

[Bei] A. Beilinson, *Localization of representation of reductive Lie algebras*, Proceedings of the International Congress of Mathematicians, Vol. 1,2 (Warsaw, 1983), PWN, Warsaw (1984), 699-710.

[BB81] A. Beilinson, J. Bernstein, *Localization de D-modules*, Comp. Rend. Acad. Sci. Paris t.292 (1981), 15-18.

[BB93] A. Bernstein, J. Bernstein, *A proof of Jantzen conjectures*, I. M. Gelfand Seminar, Adv. Soviet Math. **16**, Part 1 (1993), 1-50.

[BBD] A. Beilinson, J. Bernstein, P. Deligne, *Faisceaux pervers*, Analyse et topologie sur les espaces singulares, Astérisque **100** (1982), 1-171.

[BBM] A. Beilinson, R. Bezrukavnikov, I. Mirković, *Tilting exercises*, Mosc. Math. J. **4** (2004), no. 4, 547-557.

[BeGi] A. Beilinson, V. Ginzburg, *Wall-crossing functors and D-modules*, Represent. Theory **3** (1999), 1-31 (electronic).

[BGS] A. Beilinson, V. Ginzburg, W. Soergel, *Koszul duality patterns in Representation Theory*, Journal of A. M. S. **9** (1996), 473-527.

[BN] D. Ben-Zvi, D. Nadler, *The Character Theory of a Complex Group*, arXiv:math.RT/0904.1247v1 .

[Ber] J. Bernstein, *Trace in categories*, in *Operator algebras, unitary representations, enveloping algebras, and invariant theory*, Progress in Mathematics **92**, Birkhäuser, Paris (1989), 417-425.

[BGG73] J. Bernstein, I. M. Gelfand, S. I. Gelfand, *Schubert cells and cohomology of the spaces G/P*, Russian Math. Surveys 28 **3** (1973), 1-26.

[BGG] J. Bernstein, I. M. Gelfand, S. I. Gelfand, *A certain category of $\mathfrak{g}$-modules*, Functional Analysis and its Application **10**, no. 2, (1976), 1-8.

[BG] J. Bernstein, S. I. Gelfand, *Tensor products of finite and infinite dimensional representations of semisimple Lie algebras*, Comp. Mathematica 41, no. 2 (1980), 245-285.

[BK] A. I. Bondal, M. M. Kapranov, *Enhanced triangulated categories*, Math. USSR-Sbornik, vol. 70, no. 1, (1991), 93-107.

[Bo53] A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. (2) **57** (1953), 115-207.

[Bo91] A. Borel, *Linear algebraic groups*, Second edition, Graduate Texts in Mathematics **126**, Springer-Verlag, New York (1991).

[Bott] R. Bott, *Homogeneous vector bundles*, Ann. of Math. **66** (1957), 203-248.

[Bou68] N. Bourbaki, *Groupes et Algébres de Lie, Chapitres 4,5 et 6*, Elements de Mathématiques, Hermann, Paris (1968).

[Bou75] N. Bourbaki, *Groupes et Algébres de Lie, Chapitres 7 et 8*, Elements de Mathématiques, Hermann, Paris (1975).

[CG] N. Chriss, V. Ginzburg, *Representation theory and complex geometry*, Birkhäuser Boston, Inc., Boston, MA (1997).

[CW] A. Caldararu, S. Willerton, *The Mukai pairing, I: a categorical approach*, arXiv:math.AG/0707.2052v1.

[Ca] K. J. Carlin, *Extensions of Verma modules*, Trans. Amer. Math. Soc. **294**, no. 1 (1986), 29-43.

[ChRo] J. Chuang, R. Rouquier, *Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification*, Annals of Math. **167** (2008), 245-298.

[Donkin] S. Donkin, *The q-Schur Algebra*, London Mathematicsl Society Lecture Note Series **253**, Cambridge University Press, Cambridge (1998).

[DeWeilII] P. Deligne, *La conjecture de Weil II*, Publ. IHES **52** (1980), 137-252.

[Dem] M. Demazure, *Invariants symétriques entiers des groupes de Weyl et torsion*, Invent. Math. **21** (1973), 287-301.

[Dim] A. Dimca, *Sheaves in topology*, Universitext, Springer-Verlag, Berlin (2004).

[Dix] J. Dixmier, *Enveloping Algebras*, Graduate Studies in Mathematics **11**, American Mathematical Society, Providence, RI (1996).

[Dri] V. Drinfeld, *DG quotients of DG categories*, arXiv:math.KT/0210114v7.

[GJ] O. Gabber, A. Joseph, *Towards the Kazhdan-Lusztig conjecture*, Ann. Sci. Ecole Norm. Sup. **14** (1981), 261-302.

[GeMa] S. I. Gelfand, Y. I. Manin, *Methods of Homological Algebra*, Springer Monographs in Mathematics, New York (second edition 2003).

[Groj] I. Grojnowski, *Jantzen filtrations*, unpublished note, available at `http://www.dpmms.cam.ac.uk/~groj/papers.html`.

[Hum] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics **29**, Cambridge University Press, Cambridge (1990).

[HumCatO] J. E. Humphreys, *Representations of semisimple Lie algebras in the BGG category $\mathcal{O}$*, Graduate Studies in Mathematics **94**, American Mathematical Society, Providence, RI (2008).

[Ir] R. Irving, *Shuffled Verma modules and principal series modules over complex semisimple Lie algebras*, J. London Math. Soc. (2) **48** (1993), no. 2, 263-277.

[Ja] J. C. Jantzen, *Moduln mit einem höchsten Gewicht*, Lecture Notes in Mathematics **750**, Springer-Verlag, Berlin (1980).

[Kas] M. Kashiwara, *The Riemann-Hilbert problem for holonomic systems*, Publ. Res. Inst. Math. Sci. **20** (1984), no. 3, 319-365.

[KaSc90] M. Kashiwara, P. Schapira, *Sheaves on Manifolds*, Grundlehren der mathematischen Wissenshaften **292**, Springer-Verlag, Berlin (1990).

[KaSc06] M. Kashiwara, P. Schapira, *Categories and Sheaves*, Grundlehren der mathematischen Wissenshaften **332**, Springer-Verlag, Berlin (2006).

[KL79] D. Kazhdan, G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), no. 2, 165-184.

[KMS] M. Khovanov, V. Mazorchuk, C. Stroppel, *A brief review of abelian categorifications*, arXiv:math.RT/0702746v2.

[Kon] M. Kontsevich, *Homological algebra of Mirror Symmetry*, Proceedings of the International Congress of Mathematicians, Zürich, 1994, vol. 1, Birkhäuser (1995), 120-139.

[Lu90] G. Lusztig, *Intersection cohomology methods in representation theory*, Proceeding of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), 155-174.

[LuVo] G. Lusztig, D. A. Vogan Jr., *Singularities of closures of K-orbits on flag manifolds*, Invent. Math. **71** (1983), 365-379.

[MacL] S. MacLane, *Categories for the working mathematician*, Graduate Texts in Mathematics **5**, Springer-Verlag, New York (second edition, 1998).

[Maz03] V. Mazorchuk, *Twisted and shuffled filtrations on tilting modules*, Math. Reports of the Academy of Science of the Royal Society of Canada **25**, no. 1 (2003), 26-32.

[MOS] V. Mazorchuk, S. Ovsienko, C. Stroppel, *Quadratic duals, Koszul dual functors, and applications*, Trans. Amer. Math. Soc. **361** (2003), no. 3, 1129-1172.

[MazStr05] V. Mazorchuk, C. Stroppel, *Translation and shuffling of projectively presentable modules and a categorification of a parabolic Hecke module*, Trans. Amer. Math. Soc. **357**, no. 7 (2005), 2939-2973.

[MazStr07] V. Mazorchuk, C. Stroppel, *On functors associated to a simple root*, J. Algebra **314** (2007), no. 1, 97-128.

[MazStr08] V. Mazorchuk, C. Stroppel, *Projective-injective modules, Serre functors and symmetric algebras*, J. Reine Angew. Math. **616** (2008), 131-165.

[Mil] D. Milicic, *Localization and Representation Theory of Reductive Lie Groups*, preliminary version of a book, available at `http://www.math.utah.edu/~milicic/`.

[Neeman] A. Neeman, *Triangulated categories*, Annals of Mathematics Studies **148**, Princeton University Press, Princeton, NJ (2001).

[Ri] J. Rickard, *Translation functors and equivalences of derived categories for blocks of algebraic groups*, in "Finite dimensional algebras and related topics", Kluwer (1994), 255-264.

[RiICM] J. Rickard, *The abelian defect group conjecture*, Proceedings of the International Congress of Mathematicians, Vol. II, Berlin (1998).

[Ro] R. Rouquier, *Categorification of the braid groups*, arXiv:math.RT/0409593.

[Sa] M. Saito, *Mixed Hodge modules*, Publ. RIMS **26** (1990), 221-333.

[Sc] W. Schmid, *Vanishing theorems for Lie algebra cohomology and the cohomology of discrete subgroups of semisimple Lie groups*, Adv. in Math. **41** (1981), 77-113.

[ST] P. Seidel, R. P. Thomas, *Braid group actions on derived categories of coherent sheaves*, arXiv:math.AG/0001043v2.

[Serre] J-P. Serre, *Algèbres de Lie semi-simples complexes*, W. A. Benjamin Inc., New York-Amsterdam (1966).

[So90] W. Soergel, *Kategorie $\mathcal{O}$, perverse Garben und Moduln über den Koinvarienten zur Weylgruppe*, J. Amer. Math. Soc. **3** (1990), 421-445.

[So97] W. Soergel, *Kazhdan-Lusztig polynomials and a combinatoric for tilting modules*, Representation Theory **1** (1997), 37-68 (electronic).

[So98] W. Soergel, *Character formulas for tilting modules over Kac-Moody algebras*, Representation Theory **2** (1998), 432-448 (electronic).

[So00] W. Soergel, *On the relation between intersection cohomology and representation theory in positive characteristic*, J. Pure Appl. Algebra **152** (2000), no. 1-3, 311-335.

[So08] W. Soergel, *Andersen filtration and hard Lefschetz*, Geom. Funct. Anal. **17** (2008), no. 6, 2066-2089.

[Sp82] T. A. Springer, *Quelques applications de la cohomologie d'intersection*, Bourbaki Seminar, Vol. 1981/1982, Exp. 589, Astérisque **92-93**, Soc. Math. France, Paris, 1982.

[Sp97] , T. A. Springer, *Schubert varieties and generalizations*, Representation theories and algebraic geometry (Montreal, PQ, 1997), 412-440, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **514**, Kluwer Acad. Publ., Dordrecht (1998).

[St] C. Stroppel, *Categorification of the Temperley-Lieb category, tangles, and cobordisms via projective functors*, Duke Math. J. **126** (2005), no. 3, 547-596.

[Virk1] R. Virk, *Derived Equivalences and $\mathfrak{sl}_2$-categorifications for $U_q(\mathfrak{gl}_n)$*, preprint available at `http://www.math.wisc.edu/~virk/papers.html`.

[Virk2] R. Virk, *Derived Equivalences and Ringel self-duality of category $\mathcal{O}$*, in preparation.

[Vo] D. A. Vogan Jr., *Representations of real reductive Lie groups*, Progress in Mathematics **15**, Birkhäuser, Boston, MA (1981).