VANISHING(!) VANISHING CYCLES

Here is a statement that (to me) simultaneously sounds eminently reasonable, on closer examination looks totally false, but then turns out to be true!

Let X be a complex analytic space and K a (bounded) complex of sheaves on $X \times \mathbb{C}$. Assume that for each $x \in X$, the pullback $i_x^* K$, with $i_x : \mathbb{C} \to X \times \mathbb{C}$, $t \mapsto (x,t)$, has cohomology sheaves that are constant. Then $\phi(K) = 0$, where ϕ is the vanishing cycles functor with respect to the projection $X \times \mathbb{C} \to \mathbb{C}$.

It sounds reasonable because it seems to say that if fibrewise vanishing cycles vanish, then they vanish globally. But this heuristic is false. Vanishing cycles don't commute with taking stalks (counterexample to the heuristic: on $\mathbb{C}^2$ consider the extension by zero of the constant sheaf on the diagonal minus the origin). However, cohomology sheaves being constant is much stronger and it turns out that the original statement is true. Here is a slick proof which I still find suspicious because it doesn't seem to illuminate what is going on at all - in fact, it seems to give an even stronger statement for free.

Proof. View $X \times \mathbb{C}$ as the trivial line bundle on X and consider the evident $\mathbb{C}^\times$-action. By hypothesis, K has constant cohomology sheaves on each orbit (note: the hypothesis on K is a bit stronger, and we will need the extra in a moment). In other words, K is monodromic. Now, the Fourier-Sato transform $(-)^\wedge : D_{mon}(X \times \mathbb{C}) \to D_{mon}(X \times \mathbb{C})$ is an equivalence on the derived category of monodromic sheaves. Furthermore, it is functorial for change of base pullbacks: so $(i_x^* K)^\wedge \simeq i_x^*(K^\wedge)$. This implies that the support of $K^\wedge$ is contained in $X \times \{0\}$ (the full strength of the hypothesis on K has been used here): the Fourier-Sato transform of the rank 1 constant sheaf on $\mathbb{C}$ is the skyscraper at 0 (up to shift). But by Fourier inversion this implies that K must be a pullback of a complex of sheaves on X. Hence, $\phi(K) = 0$. $\square$

Note: doesn't really have anything to do with complex analytic spaces (just working in that setting to avoid a bunch of adjectives about what a 'reasonable' topological space is; what vanishing cycles are, etc.). No constructibility or anything of that nature is used. Could have made similar statements (with the same proof) using $\mathbb{R}$ instead of $\mathbb{C}$ and replacing 'monodromic' with 'conic'. I still do not understand what is going on here (and even though I can't seem to find any mistake in the proof above, am still very suspicious of it)! Is there a direct 'elementary' proof that doesn't invoke Fourier transform?